Chapter 4

Finite element method: one dimension

4.1. Introduction

In this chapter we introduce the finite element (FE) method (FEM) for one-dimensional, linear, second-order, scalar PDEs using the variational setting built in Chapter 3. This will elucidate the key ingredients of the finite element method and provide context for a more general variational development and the extension to more general PDEs (nonlinear, higher dimensional domains, higher order PDEs, etc). In this chapter we will intentionally be informal/vague regarding the regularity trial and test spaces. A rigorous development will be deferred to later chapters.

4.2. Model problem

In this chapter we restrict our attention to a somewhat general form of a linear, second-order, scalar PDE in one spatial dimension: given a domain $\Omega := (0, L) \subset \mathbb{R}$, find $u \in \mathcal{C}^2(\Omega)$ such that

$$-\frac{d}{dx}\left(a\frac{du}{dx}\right) = f \;\text{ in } \Omega, \quad u(0) = \bar{u}, \quad \left(a\frac{du}{dx}\right)_{x=L} = \bar{Q}, \tag{4.1}$$

where $a, f \in \mathcal{F}_{\Omega\to\mathbb{R}}$ are known functions and $\bar{u}, \bar{Q} \in \mathbb{R}$ are known scalars. There is an essential BC at $x = 0$ and a natural BC at $x = L$: $\partial\Omega_D = \{0\}$ and $\partial\Omega_N = \{L\}$. We could have chosen our model problem to have essential BCs at both ends or a natural BC at $x = 0$ and an essential BC at $x = L$; however, we could not choose natural BCs at both ends as the solution would not be unique, e.g., if u is a solution to (4.1) then so is $u + C$ where $C \in \mathbb{R}$.

Despite its simplicity, the PDE in (4.1) models a number of relevant physical systems. It governs the displacement $u : \Omega \to \mathbb{R}$ (assumed small) of an elastic bar of length L with $a := EA$, where $E, A \in \mathcal{F}_{\Omega\to\mathbb{R}_{>0}}$ are the spatially varying modulus and cross-sectional area of the bar, fixed at its left end (if $\bar{u} = 0$) and subject to a tangential load intensity $f : \Omega \to \mathbb{R}$ and an applied traction $\bar{Q}$ at its right end (Figure 4.1). It also governs temperature distribution $u : \Omega \to \mathbb{R}$ in a heat conducting bar with thermal conductivity along its length $a : \Omega \to \mathbb{R}_{>0}$, subject to a distributed heat source $f : \Omega \to \mathbb{R}$ along its length, a fixed temperature $\bar{u}$ at its left end, and imposed heat flow $\bar{Q}$ at its right end (adiabatic if $\bar{Q} = 0$). It also governs:

- flow through porous medium (fluid head u, permeability a, infiltration f, point source $\bar{Q}$),
- flow through pipes (pressure u, pipe resistance a, $f = 0$, point source $\bar{Q}$),
- flow of viscous fluids (x-velocity u, viscosity a, pressure gradient f, shear source $\bar{Q}$),
- elastic cables (displacement u, tension a, traverse force f, point source $\bar{Q}$),
- torsion of bars (angle of twist u, shear stiffness a, $f = 0$, torque $\bar{Q}$), and

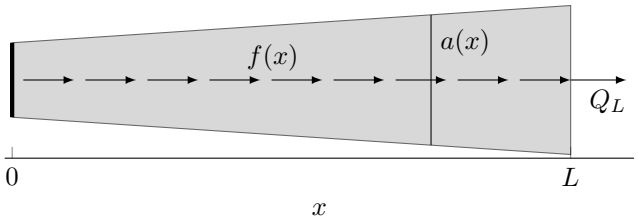

Figure 4.1: Elastic bar with axial stiffness $a(x)$ fixed at the left end subject to distributed axial load $f(x)$ and point load $\bar{Q}$.

- electrostatics (electrical potential u, dielectric constant a, charge density f, electric flux $\bar{Q}$).

From Chapter 2 the weak formulation of (4.1) is: find $u \in \mathcal{V}$ such that

$$\int_0^L \left[\frac{dw}{dx} a \frac{du}{dx} - wf \right] dx - w(L)\bar{Q} = 0 \tag{4.2}$$

for all $w \in \mathcal{W}$, where $\mathcal{V}$ is the (affine) trial space and $\mathcal{W}$ is the (linear) test space

$$\mathcal{V} := \{v \in \mathcal{F}_{\Omega \to \mathbb{R}} \mid v(0) = \bar{u}\}, \quad \mathcal{W} := \{w \in \mathcal{F}_{\Omega \to \mathbb{R}} \mid w(0) = 0\}. \tag{4.3}$$

For brevity, we will make use of the abstract bilinear form

$$B(w, u) = \ell(w), \tag{4.4}$$

where $B : \mathcal{W} \times \mathcal{V} \to \mathbb{R}$ and $\ell : \mathcal{W} \to \mathbb{R}$ are functionals

$$B(w, u) := \int_0^L \frac{dw}{dx} a \frac{du}{dx} \, dx, \qquad \ell(w) := w(L)\bar{Q} + \int_0^L wf \, dx. \tag{4.5}$$

4.3. Finite element method: formulation

The finite element method is a Ritz method in that it approximates the weak formulation of the PDE in a finite-dimensional trial and test (Galerkin) space of the form

$$\mathcal{V}_h := \varphi_h + \mathcal{V}_h^0, \quad \mathcal{W}_h := \mathcal{V}_h^0, \tag{4.6}$$

where φ_h is a affine offset satisfying the essential BC of (4.1) and $\mathcal{V}_h^0$ is a finite-dimensional linear space of solutions satisfying the homogeneous essential BC of (4.1)

$$\varphi_h \in \{f \in \mathcal{F}_{\Omega \to \mathbb{R}} \mid f(0) = \bar{u}\}, \quad \mathcal{V}_h^0 := \{v \in \mathcal{F}_{\Omega \to \mathbb{R}} \mid v(0) = 0\}, \tag{4.7}$$

where $\dim \mathcal{V}_h^0 < \infty$. That is, the finite element method approximates the weak formulation of the PDE (4.4) as: find $u_h \in \mathcal{V}_h$ such that

$$B(w_h, u_h) = \ell(w_h) \tag{4.8}$$

for all $w_h \in \mathcal{V}_h^0$. This is an approximation (not equivalent) to the weak form in (4.4) because $\mathcal{V}_h$ and $\mathcal{V}_h^0$ are proper (finite-dimensional) subsets of the trial $\mathcal{V}$ and test space $\mathcal{W}$, i.e., the FE formulation does not test against all functions in $\mathcal{W}$, only those that lie in $\mathcal{V}_h^0$. However, the fact that these approximation spaces are finite-dimensional leads to a *computable* formulation. We will often call u_h the *finite element solution*.

To this point, the FE formulation is identical to the Ritz formulation in Section 3.6. The brilliance of the finite element method comes from a specific choice for the affine offset φ_h and linear space $\mathcal{V}_h^0$ using piecewise polynomials that conveniently define families of approximation spaces that can be *refined* or enriched (in an natural way) to provide increasingly accurate approximations to the infinite-dimensional counterpart and are amenable to computer implementation.

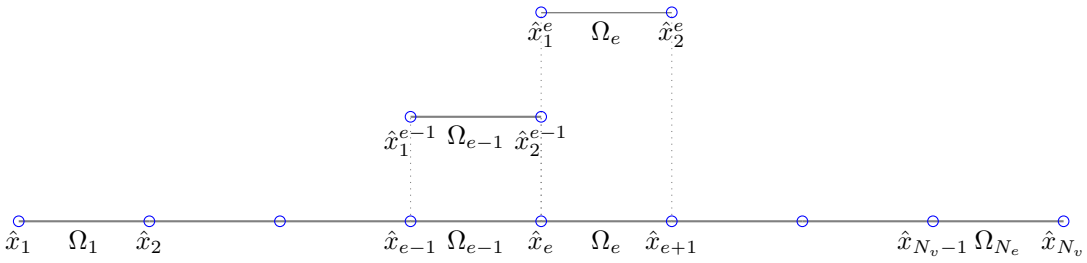

Figure 4.2: Triangulation of one-dimensional domain including global and local numbering.

To define the finite element approximation space, we begin by *discretizing* the domain Ω into $N_e := N_v - 1$ non-overlapping *finite elements*: $\mathcal{N}_h := \{\hat{x}_I\}_{I=1}^{N_v}$ is an ordered set of *nodes* such that

$$0 =: \hat{x}_1 < \hat{x}_2 < \cdots < \hat{x}_{N_v} := L, \tag{4.9}$$

with corresponding segments (elements)

$$\Omega_e := (\hat{x}_e, \hat{x}_{e+1}) \tag{4.10}$$

for $e = 1, \ldots, N_e$ (Figure 4.2). We call the collection of elements a *mesh* or *triangulation* of the domain, denoted

$$\mathcal{E}_h := \{\Omega_e\}_{e=1}^{N_e}. \tag{4.11}$$

Both the set of nodes $\mathcal{N}_h$ and elements $\mathcal{E}_h$ are ordered sets, i.e., any member of $\mathcal{N}_h$ has an associated node number, called a *global node number*, and any member of $\mathcal{E}_h$ has an associated element number. We use the notation $\hat{x}_i \in \mathcal{N}_h$ to denote the ith global node and $\Omega_e \in \mathcal{E}_h$ to denote the eth element. The length of each segment is denoted $h_e := \hat{x}_e - \hat{x}_{e-1}$ and $h := \max\{h_1, \ldots, h_N\}$ is the *mesh size parameter*, a measure of the fineness of the mesh.

For convenience we also introduce notation for the ordered set of nodes associated with element Ω_e: $\mathcal{N}_h^e := \{\hat{x}_i^e\}_{i=1}^2$ where the ordering of the members is called the *local node number* (Figure 4.2). In the simple case of a one-dimensional PDE, the global and local node numbers are related as

$$\hat{x}_e = \hat{x}_1^e, \quad \hat{x}_{e+1} = \hat{x}_2^e \tag{4.12}$$

for $e = 1, \ldots, N_e$, assuming the global/local numbering in Figure 4.2. Following the approach in Chapter 1 we describe the relationship between the global and local node numbering using the *connectivity* matrix $\Theta \in \mathbb{N}^{2 \times N_e}$,

$$\Theta = \begin{bmatrix} 1 & 2 & \cdots & N_e \\ 2 & 3 & \cdots & N_e + 1 \end{bmatrix}, \tag{4.13}$$

which abstracts (4.12) to

$$\hat{x}_j^e = \hat{x}_{\Theta_{je}}. \tag{4.14}$$

Since we are considering a *scalar* PDE, i.e., the solution is a scalar-valued function $u \in \mathcal{F}_{\Omega \to \mathbb{R}}$, there will be a single *degree of freedom* per node and therefore the local-to-global degree of freedom mapping is $\Xi = \Theta$.

Example 4.1: Mesh of one-dimensional domain

For a mesh of a one-dimensional domain $\Omega \subset \mathbb{R}$ with $N_e = 4$ elements ($N_v = 5$ nodes), we have the (ordered) node and element sets

$$\mathcal{N}_h := \{\hat{x}_1, \hat{x}_2, \hat{x}_3, \hat{x}_4, \hat{x}_5\}, \qquad \mathcal{E}_h := \{(\hat{x}_1, \hat{x}_2), (\hat{x}_2, \hat{x}_3), (\hat{x}_3, \hat{x}_4), (\hat{x}_4, \hat{x}_5)\}. \tag{4.15}$$

The local (ordered) node sets are

$$\mathcal{N}_h^1 := \{\hat{x}_1, \hat{x}_2\}, \quad \mathcal{N}_h^2 := \{\hat{x}_2, \hat{x}_3\}, \quad \mathcal{N}_h^3 := \{\hat{x}_3, \hat{x}_4\}, \quad \mathcal{N}_h^4 := \{\hat{x}_4, \hat{x}_5\}, \tag{4.16}$$

which leads to the connectivity matrix

$$\Theta = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \end{bmatrix}. \tag{4.17}$$

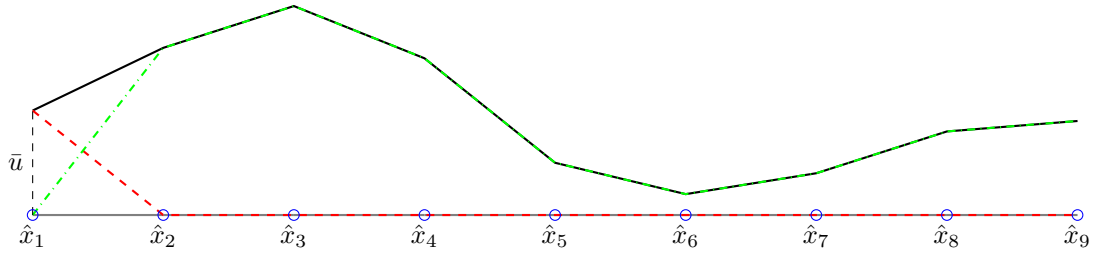

Figure 4.3: An example of an affine offset φ_h (- - -), an element of $\mathcal{V}_h^0$ (-·-·-), and an element of the trial space $\mathcal{V}_h = \varphi_h + \mathcal{V}_h^0$ (——) for a mesh with $N_e = 8$ elements.

From this concept of a mesh, we define the corresponding affine offset and *finite element subspace* (associated with the PDE in (4.1)) to be

$$\begin{aligned} \varphi_h &\in \left\{ u \in \mathcal{C}^0(\Omega) \;\middle|\; u|_{\Omega_e} \in \mathcal{P}^1(\Omega_e),\ u(0) = \bar{u} \right\} \\ \mathcal{V}_h^0 &:= \left\{ v \in \mathcal{C}^0(\Omega) \;\middle|\; v|_{\Omega_e} \in \mathcal{P}^1(\Omega_e),\ v(0) = 0 \right\}. \end{aligned} \tag{4.18}$$

The finite element subspace $\mathcal{V}_h^0$ consists of continuous functions over Ω that are linear when restricted to an element of the triangulation $\mathcal{E}_h$ and satisfy the homogeneous essential BC of (4.1). From inspection, it is clear that the dimension of this space is $\dim \mathcal{V}_h^0 = N_v - 1$ because any continuous, piecewise linear function is uniquely determined by its value at the interface between linear segments (the N_v nodes in our case); however, the boundary constraint fixes the value at $x = 0$, reducing the dimension to $N_v - 1$. The affine offset φ_h is also piecewise linear and continuous, but satisfies the non-homogeneous essential BC of (4.1). This leads to a trial space and test space of the form (4.6) that satisfy the appropriate BCs (Figure 4.3). The requirement that functions in $\mathcal{V}_h^0$ be continuous is reasonable given solutions to (4.1) will be continuous functions and is required for the FE formulation to be computable as we will see in subsequent chapters.

Remark 4.1. The use of piecewise *linear* functions to define $\mathcal{V}_h^0$ is predicated on using the weak formulation of the second-order PDE where a derivative has been moved onto the test function. The weighted residual formulation would involve a second-order differential operator that would map a piecewise linear function to zero, leading to a useless finite-dimensional formulation.

The finite-dimensional approximation of the weak form in (4.4) combined with the choices for the affine offset φ_h and finite element subspace $\mathcal{V}_h^0$ (4.18) complete the abstract formulation of the finite element method.

4.4. Construction of the finite element subspace

In this section we construct the most well-known, recognizable FE basis consisting of *nodal hat functions*. We consider two approaches to contruct the basis functions. The first method directly constructs the piecewise polynomial (linear in this case) basis functions over the triangulation $\mathcal{E}_h$, while the second approach builds first builds a *polynomial* basis for a single element and uses this to define the piecewise polynomial basis over the entire domain. The direct approach is an instructive tool and useful for analysis; however, it is not practical from an implementation viewpoint. On the other hand, the element approach is amenable to computer implementation since it never requires the entire piecewise polynomial basis functions to be formed.

4.4.1 Direct construction

To begin the direct construction of the finite element subspace $\mathcal{V}_h^0$ defined in (4.18), we first construct a collection of functions $\mathcal{A} := \{\Psi_1, \Psi_2, \ldots, \Psi_{N_v}\} \subset \mathcal{F}_{\Omega \to \mathbb{R}}$, a subset of which will be used to define a basis of the $(N_v - 1)$-dimensional space $\mathcal{V}_h^0$. We require the functions of $\mathcal{A}$ possess three key properties that will be extremely useful when it comes to an efficient computer implementation of the finite element method:

1) the functions are *nodal*, i.e., each node $I = 1, \dots, N_v$ has an associated basis function Ψ_I that possesses the *Lagrangian* property
$$\Psi_I(\hat{x}_J) = \delta_{IJ} \tag{4.19}$$
for $J = 1, \dots, N_v$,

2) the functions are *linear* when restricted to an element of $\mathcal{E}_h$, i.e., for any $\Omega_e \in \mathcal{E}_h$
$$\Psi_I|_{\Omega_e} \in \mathcal{P}^1(\Omega_e) \tag{4.20}$$
for $I = 1, \dots, N_v$, and

3) the basis function associated with node I is only non-zero on elements connected to node I, i.e., the basis functions have *local* support
$$\operatorname{supp} \Psi_1 = \overline{\Omega}_1, \qquad \operatorname{supp} \Psi_{N_v} = \overline{\Omega}_{N_v - 1}, \qquad \operatorname{supp} \Psi_I = \overline{\Omega_{I-1} \cup \Omega_I}, \quad I = 2, \dots, N_v - 1. \tag{4.21}$$

The first property implies $\mathcal{A}$ is linearly independent (Proposition 4.1). It also provides a strong connection between values of a function in $\operatorname{span} \mathcal{A}$ and its expansion in the basis $\mathcal{A}$. To see this take $f \in \operatorname{span} \mathcal{A}$, which we write as
$$f = \sum_{I=1}^{N_v} \hat{f}_I \Psi_I, \tag{4.22}$$
where $\hat{f}_1, \dots, \hat{f}_{N_v} \in \mathbb{R}$ are the coefficients defining the expansion of f in the basis $\mathcal{A}$. This is an equality between functions and must hold for any $x \in \Omega$. Evaluating at a node $x = \hat{x}_J$, we have
$$f(\hat{x}_J) = \sum_{I=1}^{N_v} \hat{f}_I \Psi_I(\hat{x}_J) = \sum_{I=1}^{N_v} \hat{f}_I \delta_{IJ} = \hat{f}_J, \tag{4.23}$$
which shows that the value of the function at node J is equal to the coefficient associated with the basis function Ψ_J, i.e., the coefficients are the *nodal values* of the function (intuition behind the *nodal* terminology). The second property implies that $\mathcal{V}_h^0 \subset \operatorname{span} \mathcal{A}$. Finally, the last property leads to *sparsity* of the finite element stiffness matrix, which is of extreme practical importance as we will see in later chapters.

Proposition 4.1 (Linear independence of nodal functions)**.** *Consider the domain $\Omega \subset \mathbb{R}^d$ with corresponding triangulation $\mathcal{E}_h$ and node set $\mathcal{N}_h$ ($N_v = |\mathcal{N}_h|$). Let $\mathcal{A} = \{\Psi_1, \dots, \Psi_{N_v}\} \subset \mathcal{F}_{\Omega \to \mathbb{R}}$ be a collection of functions with the nodal property $\Psi_I(\hat{x}_J) = \delta_{IJ}$ for $I = 1, \dots, N_v$ and $\hat{x}_J \in \mathcal{N}_h$. Then the set $\mathcal{A}$ is linearly independent.*

Proof. Take $\alpha_1, \dots, \alpha_{N_v} \in \mathbb{R}$ such that
$$\sum_{I=1}^{N_v} \alpha_I \Psi_I = 0. \tag{4.24}$$
Since this is an equation involving functions, it must hold for any $x \in \Omega$. Evaluating this equation at a node $x = \hat{x}_J$, we have
$$\sum_{I=1}^{N_v} \alpha_I \Psi_I(\hat{x}_J) = \sum_{I=1}^{N_v} \alpha_I \delta_{IJ} = \alpha_J = 0, \tag{4.25}$$
which holds for all nodes $J = 1, \dots, N_v$. Therefore the functions in $\mathcal{A}$ are linearly independent. □

The collection of functions over the one-dimensional domain $(0, L)$ satisfying these properties are the nodal hat functions, defined as
$$\Psi_I(x) := \begin{cases} \dfrac{x - \hat{x}_{I-1}}{h_{I-1}} & x \in \Omega_{I-1} \\ \dfrac{\hat{x}_{I+1} - x}{h_I} & x \in \Omega_I \\ 0 & x \notin \overline{\Omega_{I-1} \cup \Omega_I} \end{cases} \tag{4.26}$$

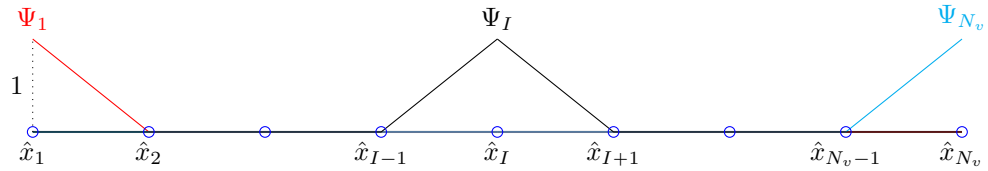

Figure 4.4: Hat function for interior node I (——) and for boundary nodes 1 (——) and N_v (——). The functions are only non-zero in elements connected to the particular node and take the value of 1 at their own node and 0 at all other nodes.

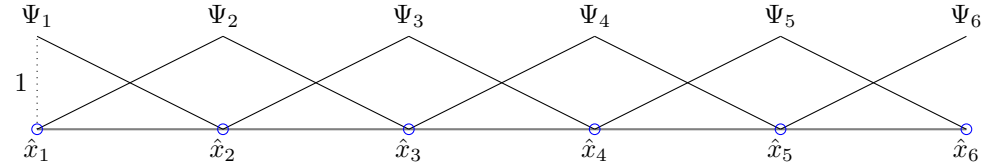

Figure 4.5: Hat functions for a complete triangulation ($N_e = 5$) of a one-dimensional domain $\Omega \subset \mathbb{R}$.

for interior nodes $I = 2, \ldots, N_v - 1$ and

$$\Psi_1(x) := \begin{cases} \dfrac{\hat{x}_2 - x}{h_1} & x \in \Omega_1 \\ 0 & x \notin \Omega_1, \end{cases} \qquad \Psi_{N_v}(x) := \begin{cases} \dfrac{x - \hat{x}_{N_v-1}}{h_{N_v-1}} & x \in \Omega_{N_v-1} \\ 0 & x \notin \Omega_{N_v-1}, \end{cases} \tag{4.27}$$

for boundary nodes (Figure 4.4-4.5). It is a simple exercise to verify the hat functions satisfy our requirements (piecewise linear nodal functions with local support). In addition, the nodal hat functions are *continuous*, i.e., $\Psi_I \in \mathcal{C}^0((0, L))$, which implies any element of span $\mathcal{A}$ are continuous (linear combinations of continuous functions are continuous functions).

We use these functions to define the FE affine offset φ_h and linear subspace $\mathcal{V}_h^0$ as

$$\varphi_h := \bar{u}\Psi_1, \qquad \mathcal{V}_h^0 := \text{span}\{\Psi_2, \ldots, \Psi_{N_v}\} \tag{4.28}$$

(Figure 4.3). For these to be valid choices according to (4.18), the affine offset must be continuous and satisfy $\varphi_h(0) = \bar{u}$ and any function $v_h \in \mathcal{V}_h^0$ must be continuous and satisfy $v_h(0) = 0$. Futhermore the $N_v - 1$ vectors defining $\mathcal{V}_h^0$ must be linearly independent to span the $(N_v - 1)$-dimensional space defined in (4.18). Continuity of φ_h and elements of $\mathcal{V}_h^0$ follow directly from continuity of the nodal hat functions $\Psi_1, \ldots, \Psi_{N_v}$ as mentioned previously. Boundary condition enforcement follows directly from the Lagrangian property

$$\varphi_h(0) = \bar{u}\Psi_1(0) = \bar{u}\Psi_1(\hat{x}_1) = \bar{u} \tag{4.29}$$

and

$$v_h(0) = v_h(\hat{x}_1) = \hat{v}_2\Psi_2(\hat{x}_1) + \cdots + \hat{v}_{N_v}\Psi_{N_v}(\hat{x}_1) = 0, \tag{4.30}$$

where $v_h \in \mathcal{V}_h^0$ is expanded in the basis $\{\Psi_2, \ldots, \Psi_{N_v}\}$ with corresponding coefficients $\hat{v}_2, \ldots, \hat{v}_{N_v} \in \mathbb{R}$. The $N_v - 1$ vectors defining $\mathcal{V}_h^0$ are linearly independent owning to Proposition 4.1. Therefore, the choices in (4.28) are valid.

4.4.2 Construction from element level

Direct construction of the finite element subspace becomes cumbersome in higher dimensions and for higher polynomial degrees. Therefore we consider an alternate approach that, instead of considering the entire domain $\Omega = (0, L)$ and building up basis functions for $\mathcal{V}_h^0$ in a global sense (Section 4.4.1), constructs the finite element space *restricted to an arbitrary element* and then *fits* elements together using the continuity requirement. This simplifies the task of constructing a basis for a piecewise polynomial space over the triangulation $\mathcal{E}_h$ to constructing one for a *polynomial* space over $\Omega_e \in \mathcal{E}_h$. Unlike the direct approach, this

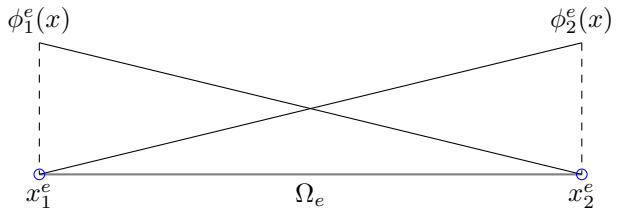

Figure 4.6: A finite element with local node numbering and $\mathcal{P}^1(\Omega)$ nodal basis.

construction will lead to a systematic procedure, amenable to computer implementation, that can naturally be generalized to higher dimensions, higher polynomial spaces, and other PDEs.

Because $\mathcal{V}_h^0$ consists of piecewise linear functions, for any $v_h \in \mathcal{V}_h^0$ we have

$$v_h|_{\Omega_e} \in \mathcal{P}^1(\Omega_e) \tag{4.31}$$

for any $\Omega_e \in \mathcal{E}_h$, where $v_h|_{\Omega_e} : \Omega_e \to \mathbb{R}$ is the restriction of v_h to Ω_e. In words this says any function of $\mathcal{V}_h^0$ is a polynomial of degree one when restricted to an element of $\mathcal{E}_h$. This implies that the restriction of v_h to Ω_e can be written

$$v_h|_{\Omega_e} = \hat{v}_1^e \phi_1^e + \hat{v}_2^e \phi_2^e, \tag{4.32}$$

where $\{\phi_1^e, \phi_2^e\}$ is a basis of $\mathcal{P}^1(\Omega_e)$ (recall $\dim \mathcal{P}^1(\Omega_e) = 2$) and $\hat{v}_1^e, \hat{v}_2^e \in \mathbb{R}$ are the corresponding coefficients. For convenience we choose a *nodal basis* that satisfies

$$\phi_i^e(\hat{x}_j^e) = \delta_{ij} \tag{4.33}$$

for $i, j = 1, 2$. Repeating the development from the previous section, the nodal property guarantees the functions $\{\phi_1^e, \phi_2^e\}$ are linearly independent and, for any function $f \in \mathcal{P}^1(\Omega_e)$ with coefficients $\hat{f}_1^e, \hat{f}_2^e \in \mathbb{R}$ in the basis $\{\phi_1^e, \phi_2^e\}$, i.e., $f = \hat{f}_1^e \phi_1^e + \hat{f}_2^e \phi_2^e$,

$$f(\hat{x}_i^e) = \sum_{j=1}^{2} \hat{f}_j^e \phi_j^e(\hat{x}_i^e) = \hat{f}_i^e. \tag{4.34}$$

In this sense, ϕ_i^e is the basis function *associated with* node $\hat{x}_i^e$ and the corresponding coefficient $\hat{f}_i^e$ is equal to the value of the function f *at that node*. The unique nodal basis of $\mathcal{P}^1(\Omega_e)$ (Figure 4.6) is

$$\phi_1^e(x) := \frac{\hat{x}_2^e - x}{h_e}, \qquad \phi_2^e(x) := \frac{x - \hat{x}_1^e}{h_e}. \tag{4.35}$$

Since $\Psi_I \in \mathcal{V}_h^0$ for $I = 1, \ldots, N_v$, its restriction to an element $\Omega_e \in \mathcal{E}_h$ can be expanded in the element basis

$$\Psi_I|_{\Omega_e} = \sum_{i=1}^{2} \hat{\alpha}_i^e \phi_i^e. \tag{4.36}$$

Furthermore since Ψ_I is a nodal function, it will take the value 1 at node I and zero at all other nodes, which leads to the condition

$$\Psi_I(\hat{x}_i^e) = \Psi_I(\hat{x}_{\Theta_{ie}}) = \delta_{I\Theta_{ie}} \tag{4.37}$$

where the first equality used the global-to-local node relationship in (4.14) and the second used the nodal property of Ψ_I. Substituting this into equation (4.36), we have

$$\delta_{I\Theta_{ie}} = \Psi_I|_{\Omega_e}(\hat{x}_i^e) = \sum_{j=1}^{2} \hat{\alpha}_j^e \phi_j^e(\hat{x}_i^e) = \hat{\alpha}_i^e, \tag{4.38}$$

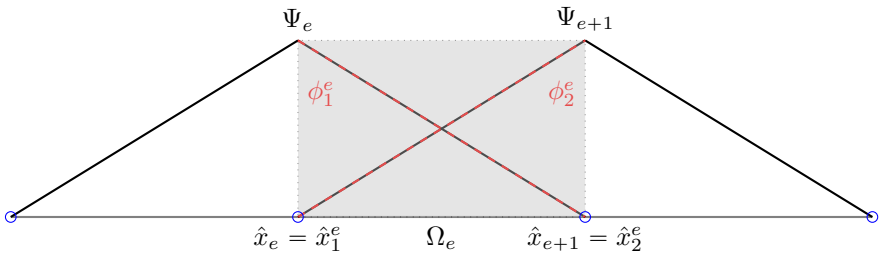

Figure 4.7: The relationship between the global finite element basis functions Ψ_I (——) and the element basis functions ϕ_i^e (- - -): the element basis functions are the restriction of the global basis functions to a single element. Shaded region: restriction to element Ω_e.

which leads to the following relationship between the element basis $\{\phi_1^e, \phi_2^e\}$ and the global functions $\{\Psi_1, \dots, \Psi_{N_v}\}$

$$\Psi_I|_{\Omega_e} = \sum_{i=1}^{2} \delta_{I\Theta_{ie}} \phi_i^e. \tag{4.39}$$

This procedure to construct the global basis functions from elementwise basis functions will naturally generalize to higher dimensions and polynomial degrees. In the special case of a scalar PDE on a one-dimensional mesh (numbering given in Figure 4.2), we use the fact that $\Theta_{1e} = e$ and $\Theta_{2e} = e + 1$ to reduce this to

$$\Psi_I|_{\Omega_e} = \delta_{Ie}\phi_1^e + \delta_{Ie'}\phi_2^e, \tag{4.40}$$

where $e' = e + 1$. For concreteness consider an interior node $I = 2, \dots, N_v - 1$, then

$$\Psi_I(x) = \begin{cases} \phi_2^I = \dfrac{x - \hat{x}_{I-1}}{h_{I-1}} & x \in \Omega_{I-1} \\ \phi_1^I = \dfrac{\hat{x}_{I+1} - x}{h_I} & x \in \Omega_I \\ 0 & x \notin \overline{\Omega_{I-1} \cup \Omega_I}, \end{cases} \tag{4.41}$$

which agrees with the expression from direct construction of Ψ_I in (4.26). Figure 4.7 illustrates this relationship between the local and global basis functions. This completes our construction of the $\mathcal{V}_h^0$ from the element level. In an implementation, we never need to explicitly form the piecewise polynoimal basis functions $\{\Psi_1, \dots, \Psi_{N_v}\}$; however, this relationship between the element and global bases will be helpful in deriving the finite element system in a way that will suggest an effective, modular computer implementation.

4.5. Finite element method

Given the construction of the finite element subspace in the previous section, we turn to deriving the finite element system. That is, introducing our subspace approximation into the finite element weak formulation to obtain an *algebraic* system of equations. Following the precedent set in the previous section, we first consider direct construction of the global stiffness matrix and load vector. While instructive this is not amenable to computer implementation in more complex scenarios. We repeat the derivation using the elementwise construction in Section 4.4.2 to obtain the global algebraic system from elementwise terms.

4.5.1 Direct construction of global system

From the construction of the nodal basis of hat functions (4.28) of the FE subspace $\mathcal{V}_h^0$ in Section 4.4.2, any element $w_h \in \mathcal{V}_h^0$ and $u_h \in \varphi_h + \mathcal{V}_h^0$ can be written as

$$w_h = \sum_{I=2}^{N_v} \hat{w}_I \Psi_I, \qquad u_h = \bar{u}\Psi_1 + \sum_{I=2}^{N_v} \hat{u}_I \Psi_I, \tag{4.42}$$

where $\hat{w}_2, \ldots, \hat{w}_{N_v} \in \mathbb{R}$ and $\hat{u}_2, \ldots, \hat{u}_{N_v} \in \mathbb{R}$ are the coefficients of w_h and u_h, respectively, in the basis $\{\Psi_2, \ldots, \Psi_{N_v}\}$. We introduce $\hat{w}_1, \hat{u}_1 \in \mathbb{R}$ so w_h and u_h can be conveniently written as

$$w_h = \sum_{I=1}^{N_v} \hat{w}_I \Psi_I, \qquad u_h = \sum_{I=1}^{N_v} \hat{u}_I \Psi_I, \tag{4.43}$$

provided $w_1 = 0$ and $u_1 = \bar{u}$; these *boundary conditions* will be imposed later. Substituting the expansions (4.43) into the FE formulation (4.8) and using linearity of the functionals yields

$$\sum_{I=1}^{N_v} \hat{w}_I \left[\sum_{J=1}^{N_v} \hat{K}_{IJ} \hat{u}_J - \hat{f}_I \right] = 0 \tag{4.44}$$

where $\hat{K} \in M_{N_v,N_v}(\mathbb{R})$ is the *global stiffness matrix* and $\hat{f} \in \mathbb{R}^{N_v}$ is the *global load vector* without considering essential boundary conditions

$$\hat{K}_{IJ} := B(\Psi_I, \Psi_J), \qquad \hat{f}_I := \ell(\Psi_I). \tag{4.45}$$

Now we impose the essential boundary conditions, i.e., require $\hat{w}_1 = 0$ and $\hat{u}_1 = \bar{u}$, to yield

$$\sum_{I=2}^{N_v} \hat{w}_I \left[\sum_{J=2}^{N_v} \hat{K}_{IJ} \hat{u}_J - (\hat{f}_I - \bar{u} \hat{K}_{I1}) \right] = 0. \tag{4.46}$$

This equation must hold for arbitrary $\hat{w}_2, \ldots, \hat{w}_{N_v} \in \mathbb{R}$ to be equivalent to the FE formulation (4.8), which holds for an arbitrary $w_h \in \mathcal{V}_h^0$. Equation (4.46) can only be true for arbitrary values of $\hat{w}_2, \ldots, \hat{w}_{N_v}$ if each term in the summation over I is zero, i.e.,

$$\sum_{J=2}^{N_v} \hat{K}_{IJ} \hat{u}_J = \hat{f}_I - \bar{u} \hat{K}_{I1} \tag{4.47}$$

for $I = 2, \ldots, N_v$, which is a (square) linear system of equations (of size $N_v - 1$) that can be solved to compute the unknown coefficients $\hat{u}_2, \ldots, \hat{u}_{N_v}$. Then the FE solution is reconstructed as

$$u_h = \bar{u}\Psi_1 + \hat{u}_2 \Psi_2 + \cdots \hat{u}_{N_v} \Psi_{N_v}. \tag{4.48}$$

To provide a strong connection to the direct stiffness method, we introduce terms that *partition* quantities based on whether or not an essential boundary condition is prescribed at the corresponding node. Let $\mathcal{I}^{\mathrm{c}}$ denote the set of node indices *constrained* by an essential BC and let $\mathcal{I}^{\mathrm{u}}$ be the *unconstrained* nodes (without essential BC). Then we use a superscript $\hat{v}^{\mathrm{u}}$ to denote the restriction of a vector over all nodes ($\hat{v} \in \mathbb{R}^{N_v}$) to only the nodes in $\mathcal{I}^{\mathrm{u}}$ nodes; $\hat{v}^{\mathrm{c}}$ is defined similarly. For matrices ($M_{N_v,N_v}(\mathbb{R})$) two superscripts are required to specify the restriction of the rows and columns, e.g., for $\hat{A} \in M_{N_v,N_v}(\mathbb{R})$, $\hat{A}^{\mathrm{cu}}$ is the restriction of rows of $\hat{A}$ to those in $\mathcal{I}^{\mathrm{c}}$ and the restriction to columns in $\mathcal{I}^{\mathrm{u}}$. In the present context, $\mathcal{I}^{\mathrm{c}} = \{1\}$ and $\mathcal{I}^{\mathrm{u}} = \{2, \ldots, N_v\}$. With this notation set, define $\hat{K}^{\mathrm{uu}} \in M_{N_v-1,N_v-1}(\mathbb{R})$, $\hat{K}^{\mathrm{uc}} \in M_{N_v-1,1}(\mathbb{R})$, $\hat{u}^{\mathrm{u}} \in \mathbb{R}^{N_v-1}$, $\hat{u}^{\mathrm{c}} \in \mathbb{R}$, and $\hat{f}^{\mathrm{u}} \in \mathbb{R}^{N_v-1}$ where

$$\hat{K}^{\mathrm{uu}}_{IJ} := \hat{K}_{I+1,J+1}, \qquad \hat{K}^{\mathrm{uc}}_{I1} := \hat{K}_{I+1,1}, \qquad \hat{u}^{\mathrm{u}}_I := \hat{u}_{I+1}, \qquad \hat{u}^{\mathrm{c}} := \bar{u}, \qquad \hat{f}^{\mathrm{u}}_I := \hat{f}_{I+1} \tag{4.49}$$

for $I, J = 1, \ldots, N_v - 1$. With these definitions (4.47) reduces to

$$\hat{K}^{\mathrm{uu}} \hat{u}^{\mathrm{u}} = \hat{f}^{\mathrm{u}} - \hat{K}^{\mathrm{uc}} \hat{u}^{\mathrm{c}}, \tag{4.50}$$

which provides a close parallel to the final system obtained using the direct stiffness method (trusses) in (1.38).

Remark 4.2. As mentioned in Section 4.3, the finite element method is a special case of the Ritz method developed in Section 3.6 using the specific subspace $\mathcal{V}_h^0$ and affine offset φ_h constructed in Section 4.4. To see this, introduce $\Phi_I := \Psi_{I+1}$ for $I = 1, \ldots, N_v - 1$ and recall that $\{\Phi_1, \ldots, \Phi_{N_v-1}\}$ was established to be a basis of $\mathcal{V}_h^0$ in Section 4.4.1. From (3.57), applying the Ritz method to the bilinear form in (4.4) with this basis leads to the stiffness matrix

$$\hat{K}^{\mathrm{ritz}}_{IJ} = B(\Phi_I, \Phi_J) = B(\Psi_{I+1}, \Psi_{J+1}) = \hat{K}_{I+1,J+1} = \hat{K}^{\mathrm{uu}}_{IJ} \tag{4.51}$$

for $I, J = 1, \ldots, N_v - 1$ and load vector

$$\hat{f}_I^{\text{ritz}} = \ell(\Phi_I) - B(\Phi_I, \varphi_h) = \ell(\Psi_{I+1}) - \bar{u}B(\Psi_{I+1}, \Psi_1) = \hat{f}_{I+1} - \bar{u}\hat{K}_{I+1,1} = \hat{f}_I^{\text{u}} - \hat{K}_{I1}^{\text{uc}}\hat{u}_1^{\text{c}} \tag{4.52}$$

for $I = 1, \ldots, N_v - 1$, which is precisely the FE system in (4.50).

We close this section by explicitly constructing the global stiffness matrix and force vector corresponding to the PDE in (4.1) based on the basis of nodal hat functions. From the definition of the stiffness matrix in (4.45) and the bilinear terms in (4.5), the entries of the stiffness matrix are

$$\hat{K}_{IJ} := B(\Psi_I, \Psi_J) = \int_0^L \frac{d\Psi_I}{dx} a \frac{d\Psi_J}{dx}\, dx \tag{4.53}$$

for $I, J = 1, \ldots, N_v$. From the expression for the hat functions we observe the integrand will only be non-zero if $|I - J| \leqslant 1$, otherwise the support of the basis functions in the integrand will not overlap and the product will be zero. This reduces the stiffness matrix to

$$\hat{K}_{IJ} = \begin{cases} \int_{\hat{x}_1}^{\hat{x}_2} \frac{d\Psi_1}{dx} a \frac{d\Psi_1}{dx}\, dx & \text{if } I = J = 1 \\ \int_{\hat{x}_{I-1}}^{\hat{x}_{I+1}} \frac{d\Psi_I}{dx} a \frac{d\Psi_I}{dx}\, dx & \text{if } 1 < I = J < N_v \\ \int_{\hat{x}_{N_v-1}}^{\hat{x}_{N_v}} \frac{d\Psi_{N_v}}{dx} a \frac{d\Psi_{N_v}}{dx}\, dx & \text{if } I = J = N_v \\ \int_{\hat{x}_{I-1}}^{\hat{x}_I} \frac{d\Psi_I}{dx} a \frac{d\Psi_{I-1}}{dx}\, dx & \text{if } J = I - 1, I > 1 \\ \int_{\hat{x}_I}^{\hat{x}_{I+1}} \frac{d\Psi_I}{dx} a \frac{d\Psi_{I+1}}{dx}\, dx & \text{if } J = I + 1, I < N_v \\ 0 & \text{otherwise} \end{cases} \tag{4.54}$$

for $I, J = 1, \ldots, N_v$. Since the nodal hat functions are piecewise linear, their derivatives are piecewise constant

$$\frac{d\Psi_I}{dx}(x) := \begin{cases} \frac{1}{h_{I-1}} & x \in \Omega_{I-1} \\ -\frac{1}{h_I} & x \in \Omega_I \\ 0 & x \notin \overline{\Omega_{I-1} \cup \Omega_I} \end{cases} \tag{4.55}$$

for interior nodes $I = 2, \ldots, N_v - 1$ and

$$\frac{d\Psi_1}{dx}(x) := \begin{cases} -\frac{1}{h_1} & x \in \Omega_1 \\ 0 & x \notin \Omega_1, \end{cases} \qquad \frac{d\Psi_{N_v}}{dx}(x) := \begin{cases} \frac{1}{h_{N_v-1}} & x \in \Omega_{N_v-1} \\ 0 & x \notin \Omega_{N_v-1}, \end{cases} \tag{4.56}$$

boundary nodes. From these equations, the stiffness matrix reduces to

$$\hat{K}_{IJ} = \begin{cases} \frac{1}{h_1^2}\int_{\hat{x}_1}^{\hat{x}_2} a\,dx & \text{if } I = J = 1 \\ \frac{1}{h_{I-1}^2}\int_{\hat{x}_{I-1}}^{\hat{x}_I} a\,dx + \frac{1}{h_I^2}\int_{\hat{x}_I}^{\hat{x}_{I+1}} a\,dx & \text{if } 1 < I = J < N_v \\ \frac{1}{h_{N_v-1}^2}\int_{\hat{x}_{N_v-1}}^{\hat{x}_{N_v}} a\,dx & \text{if } I = J = N_v \\ -\frac{1}{h_{I-1}^2}\int_{\hat{x}_{I-1}}^{\hat{x}_I} a\,dx & \text{if } J = I-1, I > 1 \\ -\frac{1}{h_I^2}\int_{\hat{x}_I}^{\hat{x}_{I+1}} a\,dx & \text{if } J = I+1, I < N_v \\ 0 & \text{otherwise.} \end{cases} \tag{4.57}$$

This expression highlights one critical reason for choosing basis functions with *local support*: the stiffness matrix is *sparse*, i.e., many of its entries are zero. Furthermore, in the special case of a scalar PDE in one dimension (with the numbering chosen according to Figure 4.2), the matrix is *tridiagonal*, i.e., of the form

$$\hat{K} = \begin{bmatrix} \times & \times & & & & \\ \times & \times & \times & & & \\ & \ddots & \ddots & \ddots & & \\ & & \times & \times & \times & \\ & & & \times & \times & \end{bmatrix} \tag{4.58}$$

where the $\times$ symbol indicates a nonzero value. This special structure means that, in addition to being very sparse, specialized algorithms exist for solving linear systems defined by $\hat{K}$ extremely efficiently using direct methods ($\mathcal{O}(N_v)$ operations, whereas direct solvers usually require $\mathcal{O}(N_v^3)$ operations).

To complete the Ritz method, we compute the load vector $\hat{f} \in \mathbb{R}^{N_v}$ as

$$\hat{f}_I := \ell(\Psi_I) = \Psi_I(L)\bar{Q} + \int_0^L \Psi_I f\,dx \tag{4.59}$$

for $I = 1, \ldots, N_v$, which can be reduced to

$$\hat{f}_I = \begin{cases} \int_{\hat{x}_1}^{\hat{x}_2} \left(\frac{\hat{x}_2 - x}{h_1}\right) f\,dx & \text{if } I = 1 \\ \int_{\hat{x}_{I-1}}^{\hat{x}_I} \left(\frac{x - \hat{x}_{I-1}}{h_{I-1}}\right) f\,dx + \int_{\hat{x}_I}^{\hat{x}_{I+1}} \left(\frac{\hat{x}_{I+1} - x}{h_I}\right) f\,dx & \text{if } 1 < I < N_v \\ \bar{Q} + \int_{\hat{x}_{N_v-1}}^{\hat{x}_{N_v}} \left(\frac{x - \hat{x}_{N_v-1}}{h_{N_v-1}}\right) f\,dx & \text{if } I = N_v \end{cases} \tag{4.60}$$

using the expressions for the hat functions (4.26)-(4.27) and their derivatives (4.55)-(4.56).

The approach to directly construct the stiffness matrix from its definition in (4.53) is striaghtforward; however, it is not practical for a number of reasons. The main reason is it relies on explicit expressions for the basis functions $\Psi_1, \ldots, \Psi_{N_v}$, which are messy to define for higher degree polynomial spaces and higher dimensional problems. In addition, it does not appear amenable to computer implementation for more complex problems due to the presence of the integrals, each of which must consider *two* elements (in general) to carry out the computation. In higher dimensions on unstructured meshes, the number of elements connected to a given node varies throughout the mesh, which makes this procedure more complicated.

4.5.2 Construction of global system from element level assembly

Now we turn to an alternative procedure to form the finite element system from element basis functions that that can be easily extended to higher dimensional space and higher degrees polynomials. To begin, we observe that the bilinear form in (4.4) can be written equivalently, owing to the additive property of integration, as a summation over element contributions

$$B(w,u) = \sum_{e=1}^{N_e} B_e(w,u), \qquad \ell(w) = \sum_{e=1}^{N_e} \ell_e(w), \tag{4.61}$$

where $B_e : \mathcal{W} \times \mathcal{V} \to \mathbb{R}$ is the restriction of the bilinear functional B to element $\Omega_e \in \mathcal{E}_h$ and $\ell_e : \mathcal{W} \to \mathbb{R}$ is the restriction of ℓ to Ω_e. In the case of the PDE in (4.5), these terms are

$$B_e(w,u) := \int_{\Omega_e} \frac{dw}{dx} a \frac{du}{dx}\, dx, \qquad \ell_e(w) := \int_{\Omega_e} wf\, dx + \begin{cases} w(L)\bar{Q} & \text{if } e = N_e \\ 0 & \text{otherwise.} \end{cases}, \tag{4.62}$$

The boundary term is only included in the element touching the boundary $x = L$ to avoid counting it multiple times. From the definition of the elementwise functionals as integrals over a single element $\Omega_e \in \mathcal{E}_h$, we have

$$B_e(w,u) = B_e(w|_{\Omega_e}, u|_{\Omega_e}), \qquad \ell_e(w) = \ell_e(w|_{\Omega_e}) \tag{4.63}$$

for any $w \in \mathcal{W}$ and $u \in \mathcal{V}$.

With these definitions, the stiffness matrix $\hat{K}$ in (4.45) can be expanded as

$$\hat{K}_{IJ} := B(\Psi_I, \Psi_J) = \sum_{e=1}^{N_e} B_e(\Psi_I, \Psi_J) = \sum_{e=1}^{N_e} B_e(\Psi_I|_{\Omega_e}, \Psi_J|_{\Omega_e}) = \sum_{e=1}^{N_e} \sum_{i=1}^{2} \sum_{j=1}^{2} B_e(\phi_i^e, \phi_j^e) \delta_{I\Theta_{ie}} \delta_{J\Theta_{je}}, \tag{4.64}$$

for $I, J = 1, \ldots, N_v$ and we used (4.61) to write the bilinear term as a summation over element contributions, (4.63) to restrict the basis functions to Ω_e, and (4.39) to relate the global basis functions to the element basis functions. Next we expand the load vector $\hat{f}$ in (4.45) as

$$\hat{f}_I := \ell(\Psi_I) = \sum_{e=1}^{N_e} \ell_e(\Psi_I) = \sum_{e=1}^{N_e} \ell_e(\Psi_I|_{\Omega_e}) = \sum_{e=1}^{N_e} \delta_{I\Theta_{ie}} \ell_e(\phi_i^e) \tag{4.65}$$

for $I = 1, \ldots, N_v$ and we used (4.61) to write the linear functional as a summation over element contributions, (4.63) to restrict the basis functions to Ω_e, and (4.39) to relate the global basis functions to the element basis functions. Finally, define the *element stiffness matrix* $\hat{K}^e \in M_{2,2}(\mathbb{R})$ and *element load vector* $\hat{f}^e \in \mathbb{R}^2$ as

$$\hat{K}^e_{ij} := B_e(\phi_i^e, \phi_j^e), \qquad \hat{f}^e_i := \ell_e(\phi_i^e), \tag{4.66}$$

which, for the bilinear form in (4.62), reduce to

$$\hat{K}^e_{ij} = \begin{cases} \frac{1}{h_e^2} \int_{x_1^e}^{x_2^e} a\, dx & i = j = 1 \\ -\frac{1}{h_e^2} \int_{x_1^e}^{x_2^e} a\, dx & i = 1, j = 2 \\ -\frac{1}{h_e^2} \int_{x_1^e}^{x_2^e} a\, dx & i = 2, j = 1 \\ \frac{1}{h_e^2} \int_{x_1^e}^{x_2^e} a\, dx & i = j = 2 \end{cases}, \qquad \hat{f}^e_i = \begin{cases} \int_{x_1^e}^{x_2^e} \left(\frac{x_2^e - x}{h_e} \right) f\, dx & i = 1 \\ \int_{x_1^e}^{x_2^e} \left(\frac{x - x_1^e}{h_e} \right) f\, dx + \bar{Q}\delta_{eN_e} & i = 2 \end{cases} \tag{4.67}$$

for $i, j = 1, 2$. The element stiffness matrix and load vector reduce the global system to

$$\hat{K}_{IJ} = \sum_{e=1}^{N_e} \sum_{i=1}^{2} \sum_{j=1}^{2} K^e_{ij} \delta_{I\Theta_{ie}} \delta_{J\Theta_{je}}, \qquad \hat{f}_I = \sum_{e=1}^{N_e} \sum_{i=1}^{2} f^e_i \delta_{I\Theta_{ie}}. \tag{4.68}$$

Notice that we have directly related the entries of the stiffness matrix and load vector to the element stiffness matrix and load vector (exactly as we did for the direct stiffness method) and the mesh connectivity Θ. Thus there is no need to form the nodal hat functions directly, saving a potentially cumbersome step.

4.6. Finite element method: implementation

With the formulation and description of the finite element method for the linear, one-dimensional model problem (4.1) complete, we turn to a systematic procedure to *implement* the finite element method that we will generalize in later chapters. The basic steps are:

(i) derive the **weak formulation** of the boundary value problem,

(ii) generate a **mesh** $\mathcal{E}_h$ for the domain Ω, including mapping the boundary conditions on the domain to the mesh,

(iii) evaluate the **element stiffness matrix** $\hat{K}^e$ and **load vector** $\hat{f}^e$ for all elements $e = 1, \dots, N_e$,

(iv) **assemble** into a global stiffness matrix and load vector, ignoring constraints coming from essential boundary conditions, and

(v) **impose** essential boundary condition.

We covered derivation of the weak formulation (i) in Chapter 3, mesh generation (ii) in one dimension is trivial (Section 4.3), and the element stiffness matrix and load vector (iii) are given in (4.62), (4.66). The assembled stiffness and load vector (iv) are given in terms of the element stiffness and load vector in (4.68). Finally, static condensation is used to impose the essential boundary conditions (v) by partitioning the finite element system into constrained and unconstrained degrees of freedom and extracting the linear system for the unconstrained degrees of freedom (4.50).

Example 4.2: Finite element method in one-dimension: Laplace equation

We would be remiss if we did not consider the standard FE example found in every FEM textbook: the Laplace equation discreted with linear finite elements on a uniform mesh. The Laplace equation is a special case of the second-order PDE in (4.1) with $a(x) = 1$, $f(x) = 0$ for all $x \in \Omega$ and we consider a unit domain $\Omega := (0, 1)$ ($L = 1$): find u such that

$$-\frac{d^2u}{dx^2} = 0 \quad \text{in } \Omega, \qquad u(0) = \bar{u}, \qquad \left(\frac{du}{dx}\right)_{x=1} = \bar{Q}, \tag{4.69}$$

where we have made a somewhat arbitrary choice of boundary conditions. We consider a mesh with $N_v = N_e + 1$ nodes numbered according to Figure 4.2 that is *uniform*, i.e., all elements the same size $|\Omega_e| = h$ for $e = 1, \dots, N_e$. For this scenario, the stiffness matrix and load vector can be directly formed using (4.57) and (4.60), respectively, to obtain

$$\hat{K} = \frac{1}{h}\begin{bmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{bmatrix} \in M_{N_v,N_v}(\mathbb{R}), \qquad \hat{f} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \bar{Q} \end{bmatrix} \in \mathbb{R}^{N_v} \tag{4.70}$$

The stiffness matrix is tridiagonal, symmetric, and positive semi-definite; however, it is not invertible since we have no imposed the essential boundary conditions yet (so there are infinitely many solutions). Alternatively, we could first form the element stiffness matrix and load vector from (4.67)

$$\hat{K}^e = \frac{1}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \qquad \hat{f}^e = \begin{bmatrix} 0 \\ \bar{Q}\delta_{eN_e} \end{bmatrix} \tag{4.71}$$

and assemble these into the appropriate positions in the global stiffness matrix using (4.68), which turns out to be a much easier, systematic procedure amenable to computer implementation. From the connectivity

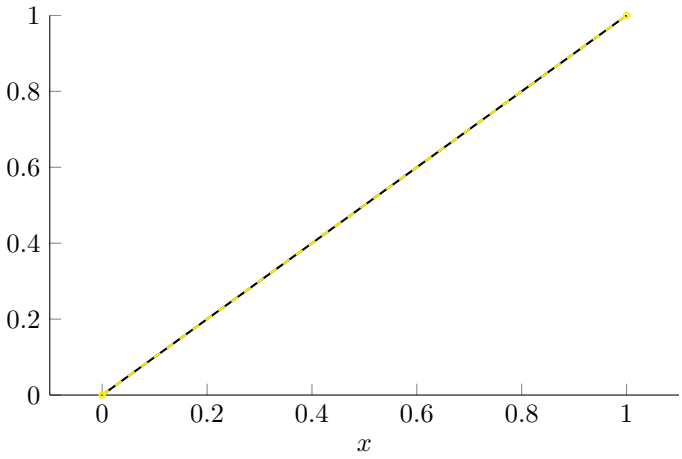

Figure 4.8: The solution to (4.69) (——) and its approximation using a single linear finite element (- ∘ -).

matrix Θ in (4.13), we have $\Theta_{ie} = e + i - 1$ so, by examining (4.68), we see that $\hat{K}^e$ will contribute to $\hat{K}_{IJ}$ for $I, J = e, e+1$. These means we will "stamp" the element matrices into the global matrix using the following pattern (special case of $N_e = 3$)

$$\hat{K} = \begin{bmatrix} \hat{K}^1_{11} & \hat{K}^1_{12} & & \\ \hat{K}^1_{21} & \hat{K}^1_{22} + \hat{K}^2_{11} & \hat{K}^2_{12} & \\ & \hat{K}^2_{21} & \hat{K}^2_{22} + \hat{K}^3_{11} & \hat{K}^3_{12} \\ & & \hat{K}^3_{12} & \hat{K}^3_{22} \end{bmatrix}. \tag{4.72}$$

It is easy to see that the global stiffness matrix in (4.70) is the result of stamping the element stiffness matrix (4.71) using this pattern. A similar stamping procedure can be identified to assemble the global load vector from the element load vectors.

To apply boundary conditions, we see the constrained set if $\mathcal{I}^{\mathrm{c}} = 1$ and the unconstrained set is $\mathcal{I}^{\mathrm{u}} = \{2, \ldots, N_v\}$, which means we remove the first column and row of $\hat{K}$ to form $\hat{K}^{\mathrm{uu}}$ and the first entry of $\hat{f}$ to form $\hat{f}^{\mathrm{u}}$

$$\hat{K}^{\mathrm{uu}} = \frac{1}{h} \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix} \in M_{N_v-1, N_v-1}(\mathbb{R}), \qquad \hat{f}^{\mathrm{u}} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \bar{Q} \end{bmatrix} \in \mathbb{R}^{N_v-1}, \tag{4.73}$$

and since the essential BC is zero ($\hat{u}^{\mathrm{c}} = \bar{u} = 0$), the second term in (4.50) vanishes. The reduced stiffness matrix (with essential BCs applied) $\hat{K}^{\mathrm{uu}}$ is tridiagonal, symmetric, and positive definite. As mentioned previously, the tridiagonal structure is important for efficiency, while positive definiteness implies the matrix is *invertible* (all positive eigenvalues) and the finite element solution is well-defined. We solve the linear system (4.50) for the unknown coefficients $\hat{u}^{\mathrm{u}} = (\hat{u}_2, \ldots, \hat{u}_{N_v})$ and form the finite element solution as

$$u_h = \hat{u}_2 \Psi_2 + \cdots \hat{u}_{N_v} \Psi_{N_v}. \tag{4.74}$$

For example take $\bar{Q} = 1$ we can compute the solution using, e.g., MATLAB, for a discretization with any number of elements $N_e = N_v - 1$. Since the solution of Laplace's equation in (4.69) is a linear function (easily determined via integration), the finite element solution is exact using only one element (Figure 4.8). A non-zero forcing term would make the solution more interesting and require a finer mesh (more elements) to sufficiently resolve.

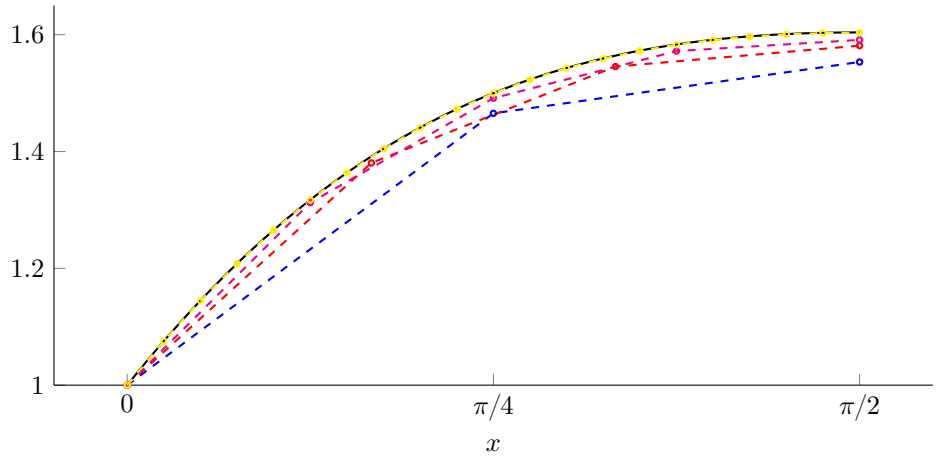

Figure 4.9: The solution to (3.8) (——) and its approximation using linear finite elements with $N_e = 2$ (- ● -), $N_e = 3$ (- ● -), $N_e = 4$ (- ● -), and $N_e = 20$ (- ● -).

Example 4.3: Finite element method in one-dimension: PDE from Chapter 3

For completeness we revisit the PDE (3.8) from Chapter 3

$$-\frac{d}{dx}\left[e^x \frac{du}{dx}(x)\right] = \sin(x) \quad \text{in } \Omega, \quad u(0) = 1, \quad \left(e^x \frac{du}{dx}\right)_{x=\pi/2} = 0, \tag{4.75}$$

over the domain $\Omega := (0, \pi/2)$ ($L = \pi/2$) whose solution we approximated with the weighted residual methods and Ritz method using $\mathcal{C}^\infty(\Omega)$ basis functions with global support. Here we use the finite element method to approximate the same problem. For simplicity, we consider a uniform mesh with N_e elements. The weak formulation of (3.8) is a bilinear form (4.4) with terms

$$B(w,u) := \int_0^{\pi/2} \frac{dw}{dx} e^x \frac{du}{dx}\, dx, \quad \ell(w) := \int_0^{\pi/2} w \sin(x)\, dx. \tag{4.76}$$

The element stiffness matrix and load vector (4.66) are

$$\hat{K}^e = \frac{1}{h^2}\begin{bmatrix} e^{x_2^e} - e^{x_1^e} & e^{x_1^e} - e^{x_2^e} \\ e^{x_1^e} - e^{x_2^e} & e^{x_2^e} - e^{x_1^e} \end{bmatrix}, \qquad \hat{f}^e = \begin{bmatrix} \cos x_1^e + \frac{\sin(x_1^e) - \sin(x_2^e)}{h} \\ -\cos x_2^e + \frac{\sin(x_2^e) - \sin(x_1^e)}{h} \end{bmatrix}, \tag{4.77}$$

which can be assembled to form a global stiffness matrix and load vector following the procedure in the previous example. To apply boundary conditions, we see the constrained set is $\mathcal{I}^{\mathrm{c}} = 1$ and the unconstrained set is $\mathcal{I}^{\mathrm{u}} = \{2, \ldots, N_v\}$, i.e.,

$$\hat{K}^{\mathrm{uu}}_{IJ} = \hat{K}_{I+1,J+1}, \quad \hat{K}^{\mathrm{uc}}_{I1} = \hat{K}_{I+1,1}, \quad \hat{f}^{\mathrm{u}}_I = \hat{f}_{I+1}, \quad \hat{u}^{\mathrm{u}}_I = \hat{u}_{I+1}, \quad \hat{u}^{\mathrm{c}}_1 = 1 \tag{4.78}$$

for $I, J = 1, \ldots, N_v - 1$. These terms define the linear system in (4.50) that we solve for the unknown $\hat{u}^{\mathrm{u}}$ and form the finite element solution as

$$u_h = \Psi_1 + \hat{u}_2 \Psi_2 + \cdots \hat{u}_{N_v} \Psi_{N_v}. \tag{4.79}$$

The finite element solution on a range of meshes is shown in the Figure 4.9 along with the exact solution. This clearly provides an improved approximation that can be systematically refined, as opposed to the methods in Chapter 3 that use basis functions with global support.

4.7. Error estimates

In this section we consider basic error analysis of the finite element method applied to the Poisson equation over the unit interval $\Omega := (0,1)$ with homogeneous essential BCs:

$$-\frac{d^2u}{dx^2} = f \quad \text{in } \Omega, \quad u(0) = u(1) = 0, \tag{4.80}$$

where $f : \Omega \to \mathbb{R}$ is given. Since the zero function is a particular solution satisfying the essential BCs, the weak formulation is: find $u \in \mathcal{V}^0$ such that

$$B(w,u) = \ell(w) \tag{4.81}$$

for all $w \in \mathcal{V}^0$ and the functionals $B : \mathcal{V}^0 \times \mathcal{V}^0 \to \mathbb{R}$ and $\ell : \mathcal{V}^0 \to \mathbb{R}$ are

$$B(w,u) := \int_\Omega \frac{dw}{dx}\frac{du}{dx}\,dx, \qquad \ell(w) := \int_\Omega wf\,dx. \tag{4.82}$$

Then the finite element formulation is: find $u_h \in \mathcal{V}_h^0$ such that

$$B(w_h,u_h) = \ell(w_h), \tag{4.83}$$

for all $w \in \mathcal{V}_h^0$, where

$$\mathcal{V}_h^0 := \left\{ v \in \mathcal{C}^0(\Omega) \;\middle|\; v|_{\mathcal{K}} \in \mathcal{P}^1(\mathcal{K}) \;\forall \mathcal{K} \in \mathcal{E}_h, v(0) = v(1) = 0 \right\} \tag{4.84}$$

and $\mathcal{E}_h$ is a mesh of Ω. It can be shown that $B(\cdot,\cdot)$ defines an inner product on the $\mathcal{V}^0$, which allows us to define the *energy norm* $|||\cdot||| : \mathcal{V}^0 \to \mathbb{R}$ as: for any $v \in \mathcal{V}^0$,

$$|||v||| := \sqrt{B(v,v)} = \left(\int_\Omega \left(\frac{dv}{dx} \right)^2 dx \right)^{1/2}. \tag{4.85}$$

The energy norm is the norm *induced* by the inner product $B(\cdot,\cdot)$. Therefore, from the properties of a norm, we have $|||v||| = 0 \implies v = 0$.

The key property to FE error estimates is *Galerkin orthogonality*: let $u \in \mathcal{V}^0$ be the solution of the weak formulation and $u_h \in \mathcal{V}_h^0$ be the solution of the finite element formulation, then

$$B(w_h, u - u_h) = 0 \quad \forall w_h \in \mathcal{V}_h^0. \tag{4.86}$$

In words, the finite element error is *orthogonal* to the space $\mathcal{V}_h^0$ in the inner product $B(\cdot,\cdot)$. This can be seen from the following argument: for any $w_h \in \mathcal{V}_h^0$, we have

$$0 = \ell(w_h) - B(w_h,u_h) = B(w_h,u) - B(w_h,u_h) = B(w_h, u - u_h), \tag{4.87}$$

where the first equality follows because u_h is the FE solution, the second equality holds because u is the solution of the weak formulation and $\mathcal{V}_h^0 \subset \mathcal{V}^0$, and the last equality follows from bilinearity. Now consider any $w_h \in \mathcal{V}_h^0$, then

$$|||u - u_h|||^2 = B(u-u_h, u-u_h) = B(u-u_h, u-w_h) + \underbrace{B(u-u_h, w_h - u_h)}_{=0 \text{ by Galerkin orthogonality}} \leqslant |||u-u_h|||\,|||u-w_h|||\,, \tag{4.88}$$

where the Cauchy-Schwartz inequality was used to obtain the final result. Dividing both sides by $|||u-u_h|||$ we see that

$$|||u-u_h||| \leqslant |||u-w_h||| \quad \forall w_h \in \mathcal{V}_h^0, \tag{4.89}$$

or, equivalently,

$$|||u-u_h||| = \inf_{w_h \in \mathcal{V}_h^0} |||u-w_h|||\,. \tag{4.90}$$

Thus the FE approximation is *optimal* in the energy norm, i.e., among all elements in $\mathcal{V}_h^0$, the FE solution is the closest to the true solution $u \in \mathcal{V}^0$. This is a significant result on its own, but also provides a means to *estimate* the error in the finite element solution. Define the interpolant of $u \in \mathcal{V}^0$ in $\mathcal{V}_h^0$ as

$$\mathcal{I}_h u := \sum_{I=1}^{N_v} u(\hat{x}_I)\Psi_I. \tag{4.91}$$

It can be shown that for any $v \in \mathcal{V}^0$ that

$$|||v - \mathcal{I}_h v||| \leqslant Ch. \tag{4.92}$$

where $C = \max_{x\in\Omega} |v''(x)|$ is a constant (assuming v'' bounded). Owing to (4.90), the FE approximation must have a *smaller* energy norm error than the interpolant because $\mathcal{I}_h u \in \mathcal{V}_h^0$, i.e.,

$$|||u - u_h||| \leqslant |||u - \mathcal{I}_h u||| \leqslant Ch, \tag{4.93}$$

which shows that the FE error decreases linearly in the energy norm as the maximum element size is decreased (in case where using piecewise linear approximation space).

4.8. Summary

This chapter introduced the finite element method in the special case of a linear, second-order partial differential equation:

1) While simple, the model problem consider can be used to model a number of different physical phenomena from deformation of a bar to flow through porous media to heat flow to electrostatics and more.

2) The finite element method is a special case of the Ritz method in that it is based on the weak formulation of a PDE and uses a Galerkin choice of finite-dimensional test/trial spaces.

3) The finite element subspace consists of *continuous*, piecewise polynomial functions (linear in this section) over a mesh, or triangulation, of the domain that satisfy the homogeneous essential boundary conditions.

4) Nodal hat functions are picewise linear functions over the domain (linear functions when restricted to any element of the mesh) associated with a particular node that possess the Lagrangian property (takes value 1 at its own node and 0 at all other nodes). They are used to define a convenient basis of the FE subspace because they are linearly independent and (a subset) span $\mathcal{V}_h^0$. The Lagrangian property implies that the coefficient associated with each function is equal to the value of the function at the node.

5) The nodal hat function do not need to be explicitly formed; rather they can be constructed from polynomial (not piecewise) bases over individual elements.

6) The algebraic form of the FE system was derived without considering essential BCs, which were later imposed via static condensation.

7) For scalar PDEs in one dimension with the natural numbering (Figure 4.2), the stiffness matrix is tridiagonal, which means it is very sparse and highly specialized solver exist to solve corresponding linear systems efficiently.

8) The five main steps of the finite element method are: (i) derive weak formulation, (ii) generate mesh of the domain, (iii) evaluate element stiffness matrix and load vector, (iv) assemble element contributions into global FE system, and (v) impose essential BCs.

9) The finite element method is optimal in the energy norm, meaning the FE solution is the point in the trial space with the smallest error as measured by the energy norm.

10) The finite element method with piecewise linear approximation spaces converges at a rate of $\mathcal{O}(h)$ in the energy norm.