AME40541/60541: Finite Element Methods
Homework 2: Due Monday, February 24, 2020

Problem 1: (10 points) Re-write the Navier equations using indicial notation and Einstein summation convention. Replace $x \rightarrow 1$, $y \rightarrow 2$, $z \rightarrow 3$.

$$\begin{aligned}
\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} + F_x = 0 \\
\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} + F_y = 0 \\
\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + F_z = 0
\end{aligned}$$

Problem 2: (15 points) (AME 60541 only) The elasticity tensor for a St. Venant-Kirchhoff material is given by $C_{ijkl} = \lambda \delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$, where λ, μ are the Lamé parameters. Calculate the stress tensor σ_{ij}, where $\sigma_{ij} = C_{ijkl}\epsilon_{kl}$ and ϵ_{kl} is the strain tensor. Make sure to use the fact that the strain tensor is symmetric ($\epsilon_{ij} = \epsilon_{ji}$). Also, calculate the deviatoric stress $s_{ij} = \sigma_{ij} - \frac{\sigma_{kk}}{3}\delta_{ij}$. In both cases, your answer should be in terms of λ, μ, and the strain tensor ϵ.

Problem 3: (10 points) From JNR 2.1: Construct the weak form of the nonlinear PDE

$$-\frac{d}{dx}\left(u\frac{du}{dx}\right) + f = 0 \;\text{ in } (0,L), \qquad \left.\left(u\frac{du}{dx}\right)\right|_{x=0} = 0, \quad u(L) = \sqrt{2},$$

for $f : (0,L) \rightarrow \mathbb{R}$ is a given function.

Problem 4: (20 points) Consider the incompressible Navier-Stokes equations that govern the flow of an incompressible fluid with density $\rho : \Omega \rightarrow \mathbb{R}_{>0}$ and viscosity $\nu : \Omega \rightarrow \mathbb{R}_{>0}$ through a domain $\Omega \subset \mathbb{R}^3$

$$\begin{aligned}
-\frac{\partial}{\partial x_1}\left(\rho\nu\frac{\partial v_1}{\partial x_1}\right) - \frac{\partial}{\partial x_2}\left(\rho\nu\frac{\partial v_1}{\partial x_2}\right) - \frac{\partial}{\partial x_3}\left(\rho\nu\frac{\partial v_1}{\partial x_3}\right) + \rho v_1\frac{\partial v_1}{\partial x_1} + \rho v_2\frac{\partial v_1}{\partial x_2} + \rho v_3\frac{\partial v_1}{\partial x_3} + \frac{\partial p}{\partial x_1} &= 0 \\
-\frac{\partial}{\partial x_1}\left(\rho\nu\frac{\partial v_2}{\partial x_1}\right) - \frac{\partial}{\partial x_2}\left(\rho\nu\frac{\partial v_2}{\partial x_2}\right) - \frac{\partial}{\partial x_3}\left(\rho\nu\frac{\partial v_2}{\partial x_3}\right) + \rho v_1\frac{\partial v_2}{\partial x_1} + \rho v_2\frac{\partial v_2}{\partial x_2} + \rho v_3\frac{\partial v_2}{\partial x_3} + \frac{\partial p}{\partial x_2} &= 0 \\
-\frac{\partial}{\partial x_1}\left(\rho\nu\frac{\partial v_3}{\partial x_1}\right) - \frac{\partial}{\partial x_2}\left(\rho\nu\frac{\partial v_3}{\partial x_2}\right) - \frac{\partial}{\partial x_3}\left(\rho\nu\frac{\partial v_3}{\partial x_3}\right) + \rho v_1\frac{\partial v_3}{\partial x_1} + \rho v_2\frac{\partial v_3}{\partial x_2} + \rho v_3\frac{\partial v_3}{\partial x_3} + \frac{\partial p}{\partial x_3} &= 0 \\
\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} &= 0
\end{aligned}$$

where $v : \Omega \rightarrow \mathbb{R}^3$ with components $v = (v_1, v_2, v_3)$ is the velocity of the fluid and $p : \Omega \rightarrow \mathbb{R}_{>0}$ is the pressure. The boundary is partitioned into two pieces: $\partial\Omega = \overline{\partial\Omega_D \cup \partial\Omega_N}$, where $n : \partial\Omega \rightarrow \mathbb{R}^3$ is the outward unit normal. The flow velocity and pressure are prescribed along $\partial\Omega_D$

$$v = \bar{v}, \;\; p = \bar{p} \quad \text{ on } \partial\Omega_D,$$

where $\bar{u} : \partial\Omega_D \rightarrow \mathbb{R}^3$ is the prescribed velocity with components $\bar{v} = (\bar{v}_1, \bar{v}_2, \bar{v}_3)$ and $\bar{p} : \partial\Omega_D \rightarrow \mathbb{R}_{>0}$ is the prescribed pressure. The traction is prescribed as $\bar{t} = (\bar{t}_1, \bar{t}_2, \bar{t}_3)$ along $\partial\Omega_N$

$$\begin{aligned}
\rho\nu\left(\frac{\partial v_1}{\partial x_1}n_1 + \frac{\partial v_1}{\partial x_2}n_2 + \frac{\partial v_1}{\partial x_3}n_3\right) - pn_1 &= \rho\bar{t}_1 \\
\rho\nu\left(\frac{\partial v_2}{\partial x_1}n_1 + \frac{\partial v_2}{\partial x_2}n_2 + \frac{\partial v_2}{\partial x_3}n_3\right) - pn_2 &= \rho\bar{t}_2 \quad \text{ on } \partial\Omega_N \\
\rho\nu\left(\frac{\partial v_3}{\partial x_1}n_1 + \frac{\partial v_3}{\partial x_2}n_2 + \frac{\partial v_3}{\partial x_3}n_3\right) - pn_3 &= \rho\bar{t}_3.
\end{aligned}$$

Re-write these equations in indicial notation and construct the weak form of the equations. Observe that in this form the equations can easily be generalized from three dimensions to d dimensions. This means you have also derived the weak formulation of the 1d, 2d (and higher dimensions!) incompressible Navier-Stokes equations as well.

Problem 5: (35 points) (AME 60541 only) Consider a system of m second-order conservation laws in a d-dimensional domain $\Omega \subset \mathbb{R}^d$

$$\nabla \cdot F(U, \nabla U) = S(U, \nabla U) \quad \text{in } \Omega,$$

where $U : \Omega \to \mathbb{R}^m$ is the state, $F(U, \nabla U) \in \mathbb{R}^{m \times d}$ is the flux function (operator), and $S(U, \nabla U) \in \mathbb{R}^m$ is a source term. The boundary conditions are $U = \bar{U}$ on $\partial\Omega_D$ and $F(U, \nabla U)n = \bar{q}$ on $\partial\Omega_N$, where $\bar{U} : \partial\Omega_D \to \mathbb{R}^m$ and $\bar{q} : \partial\Omega_N \to \mathbb{R}^m$ are known boundary functions, $\partial\Omega = \overline{\partial\Omega_D \cup \partial\Omega_N}$, and $n \in \mathbb{R}^d$ is the outward normal.

(a) (5 points) Write the conservation law in indicial notation. Drop the arguments to the flux function and source term.

(b) (10 points) Construct both the weighted residual and weak formulation of the governing equations.

(c) (5 points) What conditions must an approximate solution $U_h(x) \approx U(x)$ satisfy if applying (1) the method of weighted residuals or (2) the Ritz method? Why is it difficult to construct U_h if F is nonlinear in U or ∇U if using the method of weighted residuals?

(d) (15 points, 5 each) Write each of the following PDEs as a general conservation law, i.e., identify the state (U), flux function ($F(U, \nabla U)$), source term ($S(U, \nabla U)$), and boundary conditions ($\bar{U}$, $\bar{q}$). Also identify the PDE as linear or nonlinear (justify your answer) and the number of solution components (m).

- The second-order, linear elliptic PDE over the domain $\Omega \subset \mathbb{R}^d$

$$(k_{ij} u_{,j})_{,i} = f \;\text{ in } \Omega, \qquad u = \bar{u} \;\text{ on } \partial\Omega_D, \qquad k_{ij} u_{,j} n_i = \bar{t} \;\text{ on } \partial\Omega_N,$$

  where $u : \Omega \to \mathbb{R}$ is the unknown solution, $k : \Omega \to M_{n,n}(\mathbb{R})$ are the elliptic coefficients, $f : \Omega \to \mathbb{R}$ is the source term, $\bar{u} : \partial\Omega_D \to \mathbb{R}$ and $\bar{t} : \partial\Omega_N \to \mathbb{R}$ are boundary conditions, $n : \partial\Omega \to \mathbb{R}^d$ is the outward normal, and the boundary is $\partial\Omega = \overline{\partial\Omega_D \cup \partial\Omega_N}$.

- The linear elasticity equations govern the deformation of a domain $\Omega \subset \mathbb{R}^d$ subject to loads $f : \Omega \to \mathbb{R}^d$

$$\sigma_{ij,j} + f_i = 0 \;\text{ in } \Omega, \qquad u_i = \bar{u}_i \;\text{ on } \partial\Omega_D, \qquad \sigma_{ij} n_j = \bar{t}_i \;\text{ on } \partial\Omega_N$$

  where $\sigma : \Omega \to M_{d,d}(\mathbb{R})$ is the stress field, $\bar{u} : \partial\Omega_D \to \mathbb{R}^d$ is the prescribed boundary displacement field and $\bar{t} : \partial\Omega_N \to \mathbb{R}^d$ is the prescribed traction, $n : \partial\Omega \to \mathbb{R}^d$ is the outward normal, and the boundary is $\partial\Omega = \overline{\partial\Omega_D \cup \partial\Omega_N}$. The stress is related to the strain field $\epsilon : \Omega \to M_{d,d}(\mathbb{R})$ using Hooke's law (linear elastic material) and the strains are assumed infinitesimal

$$\sigma_{ij} = C_{ijkl}\epsilon_{kl}, \qquad \epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}).$$

- The incompressible Navier-Stokes equations (Problem 4)

Problem 6: (40 points) Consider the equations associated with a simply supported beam and subjected to a uniform transverse load $q = q_0$:

$$\frac{d^2}{dx^2}\left(EI\frac{d^2 w}{dx^2}\right) = q_0 \quad \text{for} \quad 0 < x < L$$

$$w = EI\frac{d^2 w}{dx^2} = 0 \quad \text{at} \quad x = 0, L.$$

Take $L = 1$, $EI = 1$, and $q_0 = -1$ and approximate the solution using the following methods:

(a) the method of weighted residuals (Galerkin) using the two-term trigonometric basis $w(x) \approx w_2(x) = c_1 \sin\left(\frac{\pi x}{L}\right) + c_2 \sin\left(\frac{2\pi x}{L}\right)$,

(b) the method of weighted residuals (collocation) using the same trigonometric basis and collocation nodes $x_1 = 0.25$ and $x_2 = 0.75$,

(c) the Ritz method using the same trigonometric basis, and

(d) the Ritz method using a two-term polynomial basis ($w(x) \approx w_2(x) = c_1 x(x-L) + c_2 x^2 (x-L)^2$).

For each method, verify the solution basis satisfies the appropriate conditions. Plot the approximate solution generated by each method as well as the analytical solution. In a separate figure, plot the error of each method $e(x) = |w(x) - \tilde{w}(x)|$, where $\tilde{w}$ is the approximate solution, and the residual over the domain. Finally, quantify the error of each approximation using the L^2-norm

$$e_{L^2(\Omega)} = \int_\Omega |e(x)|^2 \, dV.$$

I recommend using some symbolic mathematics software (Maple, Mathematics, MATLAB, etc) to assist with the calculations.

Problem 7: (20 points) Derive the element stiffness matrix and load vector for the following PDE

$$\begin{aligned} -\frac{d^2u}{dx^2} - u + x^2 = 0 \\ u(0) = 0, \left(\frac{du}{dx}\right)\bigg|_{x=1} = 1. \end{aligned} \tag{1}$$

and implement in `intg_elem_stiff_load_pde0.m` (starter code with comments provided on the course webite in the Homework 2 code distribution). Assume the element domain is $\Omega^e := (x_1^e, x_2^e)$ and linear Lagrangian basis functions are used:

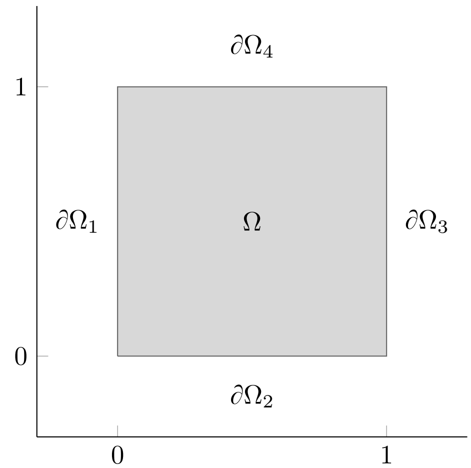
$$\phi_1^e(x) = \frac{x_2^e - x}{x_2^e - x_1^e}, \qquad \phi_2^e(x) = \frac{x - x_1^e}{x_2^e - x_1^e}.$$

Be sure to consider two cases: one that includes the boundary term and one that does not. When should the element with the boundary term included be used? As always, feel free to use any symbolic mathematics software to ease the burden of the algebra/calculus manipulations.

Problem 8: (30 points) Derive the element stiffness matrix and force vector for the following PDE over the domain $\Omega := [0,1] \times [0,1]$ (see figure below)

$$\begin{aligned} -\Delta T = 0 \quad & \text{in} \quad \Omega \\ \nabla T \cdot n = 1 \quad & \text{on} \quad \partial\Omega_1 \\ \nabla T \cdot n = 0 \quad & \text{on} \quad \partial\Omega_2 \\ T = 0 \quad & \text{on} \quad \partial\Omega_3 \cup \partial\Omega_4, \end{aligned} \tag{2}$$

and implement in `intg_elem_stiff_load_pde1.m` (starter code with comments provided on the course webite in the Homework 2 code distribution).



Square domain $\Omega = [0,1] \times [0,1]$ with boundary $\partial\Omega = \overline{\partial\Omega_1 \cup \partial\Omega_2 \cup \partial\Omega_3 \cup \partial\Omega_4}$

Assume the element domain is $\Omega^e := (x_1^e, x_2^e) \times (y_1^e, y_2^e)$ and linear Lagrangian basis functions are used:

$$\begin{aligned}
\phi_1^e(x,y) &= \left(\frac{x_2^e - x}{x_2^e - x_1^e}\right)\left(\frac{y_2^e - y}{y_2^e - y_1^e}\right) \\
\phi_2^e(x,y) &= \left(\frac{x - x_1^e}{x_2^e - x_1^e}\right)\left(\frac{y_2^e - y}{y_2^e - y_1^e}\right) \\
\phi_3^e(x,y) &= \left(\frac{x_2^e - x}{x_2^e - x_1^e}\right)\left(\frac{y - y_1^e}{y_2^e - y_1^e}\right) \\
\phi_4^e(x,y) &= \left(\frac{x - x_1^e}{x_2^e - x_1^e}\right)\left(\frac{y - y_1^e}{y_2^e - y_1^e}\right).
\end{aligned}$$

Be sure to consider two cases: one that includes the boundary term and one that does not. When should the element with the boundary term included be used? As always, feel free to use any symbolic mathematics software to ease the burden of the algebra/calculus manipulations.

Problem 9: (30 points) In this problem, you will implement a basic FEM code that we will enhance (substantially) in your final project. Before proceeding, carefully review the starter code, including all the comments, that has been provided on the course website in `hwk02−code−starter.zip`. I have provided the following functions:

- `create_mesh_hcube`: create `msh` (`xcg`, `e2vcg`) for uniform mesh of d-dimensional hypercube
- `create_ldof2gdof_cg`: create `ldof2gdof` matrix
- `assemble_nobc_mat_dense`: assemble global matrix from element matrices
- `create_dbc_strct`: create essential boundary condition structure (same as Hwk 1)
- `create_femsp_cg`: create FE space structure
- `visualize_fem`: visualize FE mesh and solution

You are welcome to use your own version of `create_ldof2gdof_cg.m` and `assemble_nobc_mat_dense.m` you implemented in Homework 1 if you made them sufficiently general.

Problem 9.1 Implement a function `eval_unassembled_stiff_load.m` that evaluates and stores the element stiffness matrix and load vector for each element in the FE mesh. Starter code is provided on the course website in the Homework 2 code distribution.

Problem 9.2 Implement a function `assemble_nobc_vec.m` that assembles the element load vector into the global load vector without applying essential boundary conditions. Starter code is provided on the course website in the Homework 2 code distribution.

Problem 9.3 Implement a function `apply_bc_solve_fem.m` that applies essential boundary conditions via static condensation to the global FE system and solves the unknown solution coefficients. Starter code is provided on the course website in the Homework 2 code distribution.

Problem 9.4 Implement a function `solve_fem_dense.m` that uses the finite element method to approximate the unknown PDE solution at nodes using the functions created in Problems 9.1-9.3. Starter code is provided on the course website in the Homework 2 code distribution.

Problem 9.5 Use the element developed in Problem 3 to approximate the solution of (1) using the finite element method. Use a mesh consisting of three linear elements and plot against the exact solution

$$u(x) = \frac{2\cos(1-x) - \sin(x)}{\cos(1)} + x^2 - 2.$$

What do you notice about the accuracy of the FEM solution at the nodes vs. interior to elements? Repeat the analysis using a finite element mesh with 25 linear elements and plot the solution.

Problem 9.6 Use the element developed in Problem 4 to approximate the solution of (2) using the finite element method. Use a mesh consisting of 3×3 linear elements and plot the solution. Repeat the analysis using a finite element mesh of 25×25 linear elements and plot the solution.