Lecture 1: Introduction

AME40541/60541: Finite Element Methods

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Outline

Modeling and simulation of physical systems

The Finite Element Method (FEM)

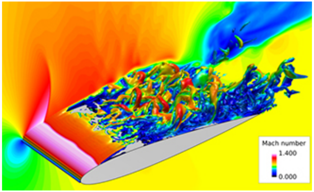
Anatomy of the finite element method

Modeling and simulation of physical systems

Fundamental goal of <u>science</u> and *engineering* is to <u>gather</u> and *apply* knowledge of physical systems

Science is primarily concerned with gaining understanding of physical systems and processes and engineering is the application of that knowledge in various contexts, e.g., design, construction, operation, maintainance.

To gain quantitative insight into such systems, mathematical models are introduced to describe their response to external input based on fundamental physical principles; these are usually *conservation laws* (mass, momentum, energy) that lead to partial differential equations.



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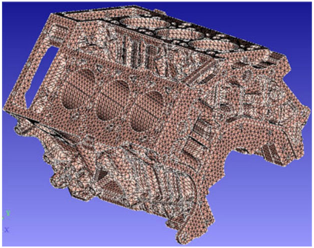

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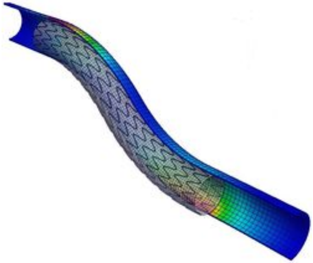

biomedical

Model of fluid dynamics: compressible Navier-Stokes equations

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) = 0,$$
$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_i}(\rho u_i u_j + p) = +\frac{\partial \tau_{ij}}{\partial x_j} \quad \text{for } i = 1, 2, 3,$$
$$\frac{\partial}{\partial t}(\rho E) + \frac{\partial}{\partial x_i}\left(u_j(\rho E + p)\right) = -\frac{\partial q_j}{\partial x_j} + \frac{\partial}{\partial x_j}(u_j \tau_{ij}),$$

Other mathematical models

- Heat flow
- Structural deformation
- Fluid-structure interaction

- Combustion
- Many, many more

Analytical methods for solving PDEs

The solution of PDEs using analytical means is limited to a small class of problems on simple domains. Analytical methods include

- separation of variables (Poisson)
- method of characteristics (1D hyperbolic)
- Fourier transform (heat flow)

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<u>Example: Separation of variables, heat equation</u>

$$\frac{\partial u}{\partial t} - \alpha \frac{\partial^2 u}{\partial x^2} = 0 \quad \text{in} \quad 0 < x < L, \qquad u(0,t) = u(L,t) = 0$$

Assume $u(x,t) = X(x)T(t)$ and substitute into original equation to obtain

$$\frac{T'(t)}{\alpha T(t)} = \frac{X''(x)}{X(x)}.$$

Since the left side only depends on t and the right side only depends on x, they must be equal to a constant $-\lambda$

$$T'(t) = -\lambda \alpha T(t), \qquad X''(x) = -\lambda X(x), \tag{1}$$

which are linear ODEs that can be solved using standard methods.

Analytical methods for solving PDEs are rarely an option

However, practical problems that arise in science and engineering rarely fit into the small class of problems that can be solved analytically, either because the PDEs are complicated or must be solved on complex domains.

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \nu \Delta \boldsymbol{u} = 0$$
$$\nabla \cdot \boldsymbol{u} = 0$$

The Clay Mathematics Institute has identified seven classic questions that have resisted solution for many years and offers a $1M award for the solution of each (Millennium Prizes). Proof of existence and smoothness of solutions of the Navier-Stokes equations (developed in 1822 and currently the most widely accepted model of fluid dynamics) is one of the seven problems.

Numerical methods and powerful computers to approximate solutions of PDEs

- Solutions of PDEs are (vector-valued) functions of space and time, e.g., the displacement of a structure at each material point for all time, that live in infinite dimensional function spaces.
- Elements of function spaces cannot directly be represented in a computer since they have finite memory and only encode finite-precision numbers and operations on them.
- *Numerical methods* are a means to convert the PDE, equations over functions spaces, to a discrete (algebraic) system of equations, a process called **discretization**.
- The most popular numerical methods are the Finite Difference Method (FDM), Finite Volume Method (FVM), and the **Finite Element Method (FEM)**.

The Finite Element Method (FEM)

What is the finite element method?

The finite element method is a numerical method for discretizing PDEs that:

- employs a variational setting (weak formulation of the PDE),
- requires a computational mesh, a discretization of the domain and boundary into a collection of elements of known shape, and
- approximates the solution in a continuous basis where each basis function has local support over a single element.

It is provably optimal for elliptic PDEs and therefore the **industry-standard** for structural and thermal problems; however, it is not stable for advection-dominant problems (high-speed flows). Many variants of the finite element method, including streamwise-upwind Petrov-Galerkin and discontinuous Galerkin methods, have resolved the stability issues of finite element methods and they are becoming relevant for flow problems, particularly when high-order approximations are desired.

The finite element method and its competitors

Finite Element Method

- Approximates the variational form of the PDE in a finite-dimensional space
- Handles arbitrarily complex domains
- Optimal for elliptic problems
- High-order approximations readily available
- Unstable for advection-dominant problems (unless variant used)

Finite Difference Method

- Directly approximates the strong formulation of the PDE, replacing each term with its finite difference approximation
- Simple to understand and implement
- Only works for the simplest domains

Finite Volume Method

- Recasts the PDE into one that governs the behavior of cell-wise averages
- Handles arbitrarily complex domains
- Workhorse for flow problems due to stability properties
- Difficult to implement and extend to a high-order approximation

Anatomy of the finite element method

Problem description and transformation

Governing equations, boundary conditions, and domain

- Find the solution, $u(x)$, that satisfies the partial differential equation

$$-\frac{d}{dx}\left[\beta(x)\frac{du}{dx}\right] = f(x), \quad a < x < b$$

with boundary conditions $u(a) = u_a, u(b) = u_b$.

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Derive variational (weak) form of the equations

- It can be shown (and we will!) that $u(x)$ is a solution of the PDE if and only if it is a solution of the variational equation

$$\int_a^b w(x)\left[-\frac{d}{dx}\left[\beta(x)\frac{du}{dx}\right] - f(x)\right] dx = 0$$

  for all $w(x)$ such that $w(a) = w(b) = 0$
- Perform integration-by-parts to transfer a derivative from u to w to weaken the regularity requirements on the solution approximation

$$-\int_a^b \beta(x)\frac{dw}{dx}\frac{du}{dx}\,dx = \int_a^b w(x)f(x)\,dx$$

Numerical approximation and solution

Discretize domain into nodes and elements

Partition domain into finite elements and introduce a *local* basis for the solution and test functions

$$u(x) \approx u^h(x) = \sum_{I=1}^{N} u_I^h \phi_I(x), \qquad w(x) \approx w^h(x) = \sum_{I=1}^{N} w_I^h \phi_I(x)$$

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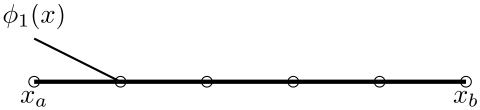

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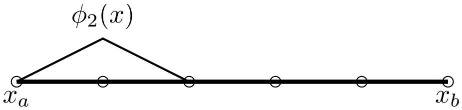

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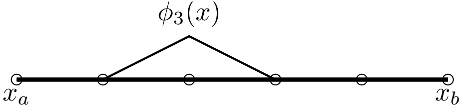

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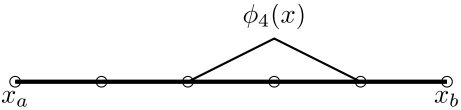

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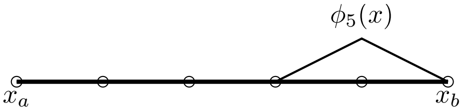

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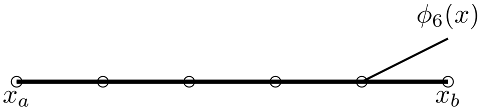

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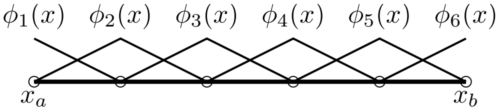

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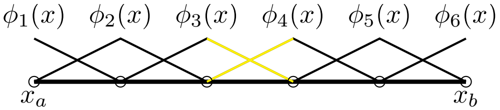

Break govering equation into element contributions

$$\sum_{e=1}^{N_e} -\int_{x_a^e}^{x_b^e} \beta(x) \frac{dw^h}{dx} \frac{du^h}{dx}\, dx = \sum_{e=1}^{N_e} \int_{x_a^e}^{x_b^e} w^h(x) f(x)\, dx$$

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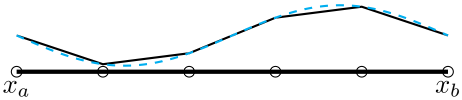

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Assemble element contributions into global system and solve

$$\boldsymbol{K}\boldsymbol{u} = \boldsymbol{F}$$