AME40541/60541: Finite Element Methods
Homework 2: Due Monday, February 15, 2021

Problem 1: (30 points) Consider Truss 0 in Figure 1 and assume the Young's modulus times the cross-sectional area for each element is $EA_e = e$ for $e = 1, \ldots, 5$.

- (10 points) What is the element stiffness matrix K_{ij}^e for elements 1 and 5?
- (10 points) Identify all entries in the global stiffness matrix to which 3 contributes, i.e., which entries in $\boldsymbol{K}$ does $\boldsymbol{K}^3$ contribute, where $\boldsymbol{K}$ is the global stiffness matrix. What about element 5?
- (10 points) What are K_{11}, K_{34}, K_{65}, K_{77}, K_{78}, K_{87}, K_{88} in terms of the element stiffness matrices K_{ij}^e?

Problem 2: (20 points) (AME60541 only) Consider a truss structure with elastic boundary conditions, e.g., Truss 1 (Figure 1). The spring is at rest when the truss is in its undeformed configuration. Recall the force in a spring is $F = k\Delta x$ where Δx is the deformation of the spring from its rest configuration. How does the direct stiffness method change when considering elastic boundary conditions? Provide a description in the context of Truss 1 then generalize the procedure to a general 2d truss structure with an elastic boundary condition on any global degree of freedom.

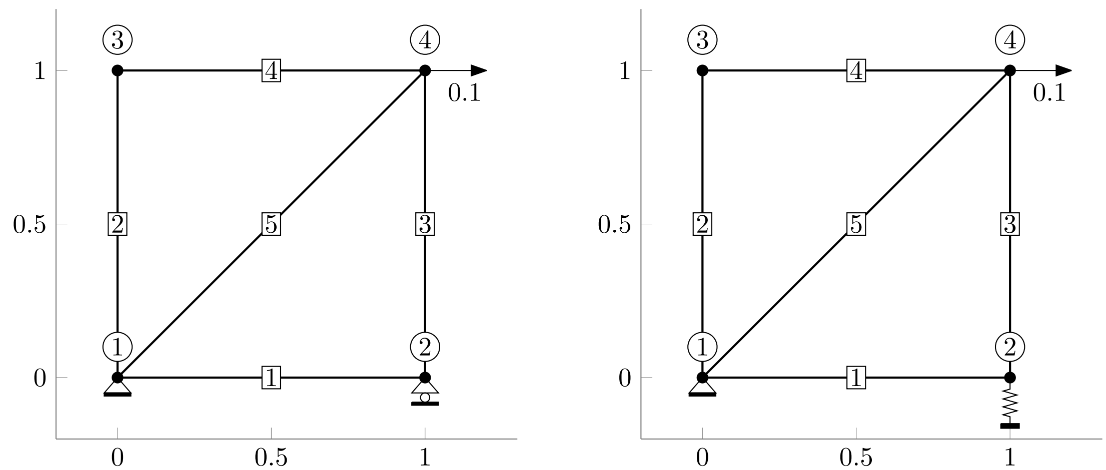
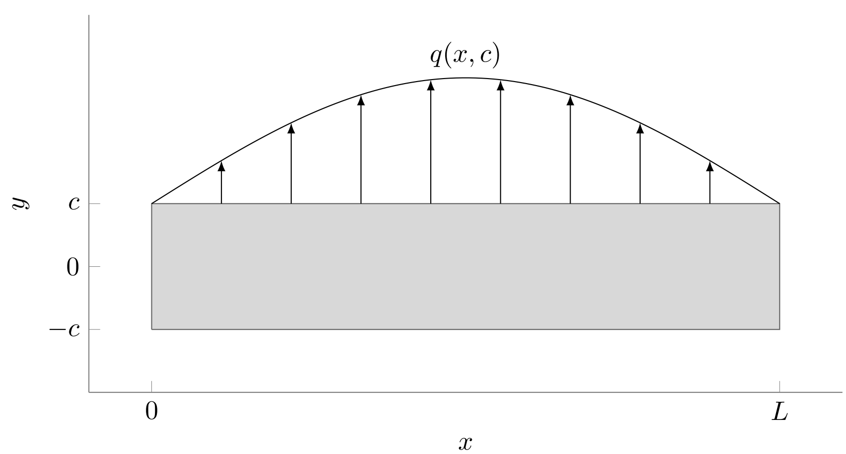


Figure 1: Truss 0 (*left*) and Truss 1 (*right*)

Problem 3: (30 points) From S. Govindjee, UC Berkeley. Consider the two-dimensional beam subject to a transverse sinusoidal load below.



Let $u(x,y)$ denote the x-displacement and $v(x,y)$ the y-displacement and $q(x,c)$ denote the *load per unit area*. The exact solution of the y-displacement along the centerline of the beam with the boundary conditions

$$u(0,0) = v(0,0) = v(L,0) = 0, \qquad q(x,c) = q_0 \sin\left(\frac{\pi x}{L}\right),$$

and the geometric condition $L \gg c$, is

$$v(x,0) = \frac{3q_0 L^4}{2c^3\pi^4 E} \sin\left(\frac{\pi x}{L}\right)\left[1 + \frac{1+\nu}{2}\frac{\pi c}{L}\tanh\left(\frac{\pi c}{L}\right)\right].$$

This analytical solution was derived under the plane stress assumptions. Select reasonable values for the geometry (the L/c ratio should be at least 20), material properties (stiffness E and Poisson ratio ν), and load q_0.

(a) What is the analytical solution at $(x,y) = (L/2, 0)$ for the parameters you chose?

(b) Model this system in COMSOL. Be sure to use the plane stress assumption with a thickness of $t = 0.001L$ and apply boundary conditions exactly as specified above i.e., do not fix the displacements along an entire edge. For the discretization, consider both linear triangles and quadrilateral elements on a sequence of at least four meshes each of increasing refinement. For each mesh, compute the solution and output $v(L/2, 0)$.

- Make sure to output the vertical displacement rather than the total displacement, which is the default (modify *Expression* to v, rather than *solid.disp*). If loads are small enough, difference between the total and vertical displacement will be negligible.
- Model the domain as two rectangles, one over $[0,L]\times[-c,0]$ and one over $[0,L]\times[0,c]$. Some versions of COMSOL with have trouble generating a quadrliateral mesh if an isolated point, e.g., at $(L/2,0)$, exists. Note: may not be necessary in newer versions of COMSOL.
- COMSOL does not allow you to take the sine of a number with *units*. Therefore, to prescribe the distributed load $\sin(\pi x/L)$, this must be entered as `sin(pi*(x/L[m]))` if you are working in units of meters. Also, be sure to apply the load per unit length to be consistent with the assumptions under which the analytical solution was derived.
- Be sure to specify the load per unit area (not per unit length).

(c) On a single figure, plot $v(L/2,0)$ versus the number of elements for both mesh sequences. Also include the exact solution as a horizontal line (since it does not depend on the number of elements). What do you observe about the accuracy of triangular vs. quadrilateral elements for bending problems?