AME40541/60541: Finite Element Methods
Homework 4: Due Monday, March 1, 2021

Problem 1: (10 points) Re-write the Navier equations using indicial notation and Einstein summation convention. Replace $x \to 1$, $y \to 2$, $z \to 3$.

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} + F_x = 0$$
$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} + F_y = 0$$
$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + F_z = 0$$

Problem 2: (15 points) (AME 60541 only) The elasticity tensor for a St. Venant-Kirchhoff material is given by $C_{ijkl} = \lambda \delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$, where λ, μ are the Lamé parameters. Calculate the stress tensor σ_{ij}, where $\sigma_{ij} = C_{ijkl}\epsilon_{kl}$ and ϵ_{kl} is the strain tensor. Make sure to use the fact that the strain tensor is symmetric ($\epsilon_{ij} = \epsilon_{ji}$). Also, calculate the deviatoric stress $s_{ij} = \sigma_{ij} - \frac{\sigma_{kk}}{3}\delta_{ij}$. In both cases, your answer should be in terms of λ, μ, and the strain tensor ϵ.

Problem 3: (10 points) From JNR 2.1: Construct the weak form of the nonlinear PDE

$$-\frac{d}{dx}\left(u\frac{du}{dx}\right) + f = 0 \quad \text{in } (0, L), \qquad \left.\left(u\frac{du}{dx}\right)\right|_{x=0} = 0, \quad u(L) = \sqrt{2},$$

for $f : (0, L) \to \mathbb{R}$ is a given function.

Problem 4: (30 points) The linear elasticity equations model structural deformation in the limit of infinitesimal strain and a linear relationship between stress and stress

$$\sigma_{ij,j} + f_i = 0 \quad \text{in } \Omega,$$
$$\sigma_{ij}n_j = \bar{t}_i \quad \text{on } \partial\Omega,$$

for $i = 1, \dots, d$, where the stress tensor $\boldsymbol{\sigma} \in \mathbb{R}^{d\times d}$ and strain tensor $\boldsymbol{\epsilon} \in \mathbb{R}^{d\times d}$ are defined as

$$\sigma_{ij} = C_{ijkl}\epsilon_{kl}, \qquad \epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}),$$

and $u_i \in \mathbb{R}$ is the displacement in the ith direction for $i = 1, \dots, d$. We will solely consider a homogeneous, isotropic material with $C_{ijkl}(x) = \lambda(x)\delta_{ij}\delta_{kl} + \mu(x)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$, where $\lambda(x)$ and $\mu(x)$ are the Lamé parameters. Considering the special case of $d = 2$ is equivalent to making the *plane strain* assumption.
Consider a multimaterial beam (Figure 1) with boundary conditions: clamped on $\partial\Omega_1$ ($u_1 = u_2 = 0$), no traction on $\partial\Omega_2 \cup \partial\Omega_4$ ($\bar{t}_1 = \bar{t}_2 = 0$), and a distributed force in the $-y$ direction of 0.1 on $\partial\Omega_3$ ($\bar{t}_1 = 0, \bar{t}_2 = -0.1$). Take the Lamé parameters for material 1 to be $\lambda_1(x) = 365$, $\mu_1(x) = 188$ and those for material 2 to be $\lambda_1(x) = 36.5$, $\mu_1(x) = 18.8$.

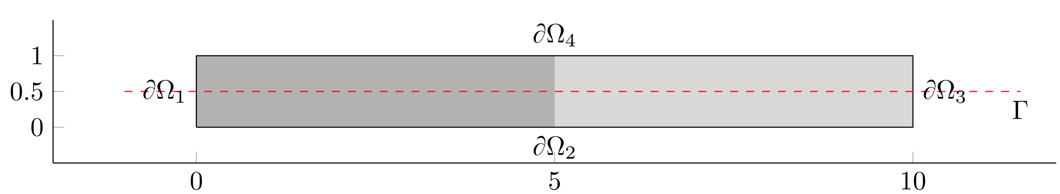

Figure 1: Multimaterial beam (Ω), boundaries ($\partial\Omega_i$), and line along which to evaluate quantities (Γ).

(a) Use COMSOL to approximate the solution to the linear elasticity equations on a sufficiently refined mesh.

(b) Evaluate the displacements u_1, u_2 along the line Γ (Figure 1) and plot the von Mises stress on the deformed geometry.

I suggest saving the numeric value of the slices as we will solve this problem using the FE code you develop during your final project; these values will provide a valuable test for your implementation.