AME40541/60541: Finite Element Methods
Homework 6: Due Monday, March 15, 2021

Problem 1: (20 points) Derive the element stiffness matrix and load vector for the following PDE

$$\begin{aligned} -\frac{d^2u}{dx^2} - u + x^2 = 0 \\ u(0) = 0, \left(\frac{du}{dx}\right)\Big|_{x=1} = 1. \end{aligned} \tag{1}$$

over the domain $\Omega = [0, 1]$, and implement in `intg_elem_stiff_load_pde0.m` (starter code with comments provided on the course webite in the Homework 6 code distribution). Assume the element domain is $\Omega^e := (x_1^e, x_2^e)$ and linear Lagrangian basis functions are used:

$$\phi_1^e(x) = \frac{x_2^e - x}{x_2^e - x_1^e}, \qquad \phi_2^e(x) = \frac{x - x_1^e}{x_2^e - x_1^e}.$$

Be sure to consider two cases: one that includes the boundary term and one that does not. When should the element with the boundary term included be used? As always, feel free to use any symbolic mathematics software to ease the burden of the algebra/calculus manipulations.

Problem 2: (30 points) Derive the element stiffness matrix and force vector for the following PDE over the domain $\Omega := [0, 1] \times [0, 1]$ (see figure below)

$$\begin{aligned} -\Delta T &= 0 \quad \text{in} \quad \Omega \\ \nabla T \cdot n &= 1 \quad \text{on} \quad \partial\Omega_1 \\ \nabla T \cdot n &= 0 \quad \text{on} \quad \partial\Omega_2 \\ T &= 0 \quad \text{on} \quad \partial\Omega_3 \cup \partial\Omega_4, \end{aligned} \tag{2}$$

and implement in `intg_elem_stiff_load_pde1.m` (starter code with comments provided on the course webite in the Homework 6 code distribution).

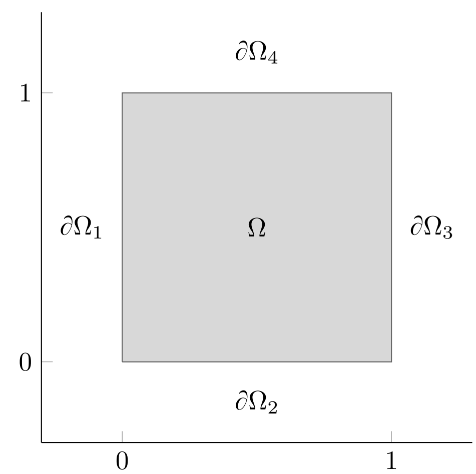

Figure 1: Square domain $\Omega = [0, 1] \times [0, 1]$ with boundary $\partial\Omega = \overline{\partial\Omega_1 \cup \partial\Omega_2 \cup \partial\Omega_3 \cup \partial\Omega_4}$.

Assume the element domain is $\Omega^e := (x_1^e, x_2^e) \times (y_1^e, y_2^e)$ and linear Lagrangian basis functions are used:

$$
\begin{aligned}
\phi_1^e(x,y) &= \left(\frac{x_2^e - x}{x_2^e - x_1^e}\right)\left(\frac{y_2^e - y}{y_2^e - y_1^e}\right) \\
\phi_2^e(x,y) &= \left(\frac{x - x_1^e}{x_2^e - x_1^e}\right)\left(\frac{y_2^e - y}{y_2^e - y_1^e}\right) \\
\phi_3^e(x,y) &= \left(\frac{x_2^e - x}{x_2^e - x_1^e}\right)\left(\frac{y - y_1^e}{y_2^e - y_1^e}\right) \\
\phi_4^e(x,y) &= \left(\frac{x - x_1^e}{x_2^e - x_1^e}\right)\left(\frac{y - y_1^e}{y_2^e - y_1^e}\right).
\end{aligned}
$$

Be sure to consider two cases: one that includes the boundary term and one that does not. When should the element with the boundary term included be used? As always, feel free to use any symbolic mathematics software to ease the burden of the algebra/calculus manipulations.