AME60714: Advanced Numerical Methods
Homework 1: Due Monday, September 14, 2020

Problem 1: (10 points) Write a function that solves the Riemann problem for a general linear system

$$q_{,t} + Aq_{,x} = 0, \quad -\infty < x < \infty,\ t > 0$$
$$q(x,0) = \begin{cases} q_l & x < 0 \\ q_r & x > 0 \end{cases}$$

given $A \in \mathbb{R}^{m\times m}$ and $q_l, q_r \in \mathbb{R}^m$. The function should sample the conservative $q(x,t)$ and characteristic $w(x,t)$ variables for all $x \in \mathcal{X} \subset \mathbb{R}$ and $t \in \mathcal{T} \subset \mathbb{R}_{\geqslant 0}$, where the sets $\mathcal{X}$ and $\mathcal{T}$ are input arguments. Then for each of the following hyperbolic systems, plot: snapshots of the conservative and characteristic variables in time, the domain of dependence and determinacy of the point $(x,t) = (0.5, 0.5)$, and the range of influence of the points $(x,t) = (0,0)$ (truncated at a final time of $t_{\max} = 0.5$).

a) $A = \begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix}, \quad q_l = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad q_r = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

b) $A = \begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix}, \quad q_l = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad q_r = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

c) $A = \begin{bmatrix} 0 & 9 \\ 1 & 0 \end{bmatrix}, \quad q_l = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad q_r = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$

d) $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad q_l = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad q_r = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

e) $A = \begin{bmatrix} 2 & 1 \\ 10^{-4} & 2 \end{bmatrix}, \quad q_l = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad q_r = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Problem 2: (20 points) Consider the general form of the scalar nonlinear conservation law

$$q_{,t} + f(q)_{,x} = 0 \tag{1}$$

with the following initial condition

$$q(x,0) = \begin{cases} \bar{q} & x \in [-a, a] \\ 0 & x \notin [-a, a] \end{cases} \tag{2}$$

for $a > 0$ and $\bar{q} > 0$. For Burgers equation ($f(q) = q^2/2$) and traffic flow ($f(q) = cq(1-q)$), complete the following tasks.

a) What type of wave (shock, rarefaction) is originating from $x = -a$ and $x = a$? Hint: check the entropy condition.

b) What is the shock speed?

c) What is the propagation speed of the left and right edge of the rarefaction?

d) What is T_c, the time the rarefaction and shock collide?

e) Take $a = 1$, $\bar{q} = 1$, $c = 1$, and plot:

- the characteristics for $x \in [-3, 3]$, $t \leqslant T_c$
- snapshots of the solution $q(x,t)$ at five equidistance times in $[0, T_c]$.

Problem 3: (30 points) (adapted from LeVeque 13.7-13.9) Consder the p-system

$$\begin{aligned} v_t - u_x &= 0, \\ u_t + p(v)_x &= 0, \end{aligned} \tag{3}$$

where $p(v)$ is a given function of v.

(a) Compute the eigenvalues and eigenvectors of the Jacobian matrix.

(b) What is the condition on $p(v)$ that must be satisfied for the system to be strictly hyperbolic?

(c) What is the condition on $p(v)$ that must be satisfied for both fields to be genuinely nonlinear for all q?

(d) Use the Rankine-Hugoniot condition to show the Hugoniot loci of states connected to some fixed state $q_* = (v_*, u_*)$ is given by $q(\xi) = (\xi, u(\xi))$, where

$$u(v) = u_* \pm \sqrt{-\left(\frac{p(v) - p(v_*)}{v - v_*}\right)}(v - v_*). \tag{4}$$

What is the propagation speed for such a shock? Plot the Hugoniot loci for the point $q_* = (1, 1)$ for $v \in [-3, 5]$ for $p(v) = -e^v$.

(e) Derive the integral curves and Riemann invariants for both fields (assume $p(v) = -e^v$).

Problem 4: (30 points) Consider the p-system in (3) with $p(v) = -e^v$.

a) Derive the Roe linearization $\tilde{A}(q_l, q_r)$ using the identity parameter vector $z(q) = q$. Identify the Roe average $\hat{q}$ by writing the final form of the Roe linearization as $\tilde{A}(q_l, q_r) = f'(\hat{q})$ (since $f'(q)$ only involves the v variable, only need to identify the Roe average for v).

b) Prove the Roe linearization is consistent, i.e., show

$$\lim_{q_r, q_l \to q} \tilde{A}(q_l, q_r) = f'(q),$$

hyperbolic (diagonalizable with real eigenvalues), and conservative at discontinuities, i.e., $f(q_r) - f(q_l) = \tilde{A}(q_l, q_r)(q_r - q_l)$.

c) Derive the numerical flux function for the HLL and Roe approximate Riemann solvers.

d) Write functions that implement the HLL and Roe approximate Riemann solvers and evaluates the corresponding numerical flux; compare to the exact Riemann solution and numerical flux (MATLAB code provided). For HLL, use the following wave speed estimates

$$s_1 = -e^{\bar{v}/2}, \quad s_2 = e^{\bar{v}/2},$$

where $\bar{v}(q_l, q_r) = (v_l + v_r)/2$ is the arithmetic average. For Roe, the Riemann solver for linear hyperbolic systems from Problem 1 can be used once the linearized Roe matrix is computed. For each of the data below with $\alpha = 1$, plot a snapshot of the u and v solution at time $t = 0.3$ using the three Riemann solvers (exact, HLL, Roe)

1) 1- and 2-rarefaction: $q_l = (1, 1 + 2\alpha)$, $q_r = (1, 1)$
2) 1-shock, 2-rarefaction: $q_l = (1, 1 + \alpha/2)$, $q_r = (1 + \alpha, 1)$
3) 1-rarefaction, 2-shock: $q_l = (1 + \alpha, 1)$, $q_r = (1, 1 + \alpha/2)$.

Comment on the differences between the approximate Riemann solvers, accuracy with respect to the exact Riemann solution, and the dependence on α (strength of jump in initial data).

Problem 5: (20 points) Consider the d-dimensional Euler equations (ideal gas law)

$$q_{,t} + \nabla \cdot f(q) = 0,$$

where the conservative variables and flux function are

$$q = \begin{bmatrix} \rho \\ \rho\vec{v} \\ \rho E \end{bmatrix}, \qquad f(q) = \begin{bmatrix} \rho\vec{v}^T \\ \rho\vec{v} \otimes \vec{v} + pI \\ \rho H \vec{v}^T \end{bmatrix}$$

and the pressure p and enthalpy H are defined as

$$p(q) = (\gamma - 1)(\rho E - \rho\vec{v} \cdot \vec{v}/2), \qquad H(q) = \frac{1}{\rho}\left(\rho E + p(q)\right)$$

a) Prove the flux function is rotationally invariant: for any rotation matrix $Q \in \mathbb{R}^{m \times m}$,

$$f(q)n = \hat{Q} f(\hat{Q}^T q) Q^T n,$$

where

$$\hat{Q} = \begin{bmatrix} 1 & & \\ & Q & \\ & & 1 \end{bmatrix}.$$

b) Derive the eigenvalue decomposition of

$$B(q, n) := \frac{\partial [f(q)n]}{\partial q}$$

for a general state $q \in \mathbb{R}^m$ and unit vector $n \in \mathbb{R}^d$ in terms of the eigenvalue decomposition of $B(q, e_1)$, where $e_1 \in \mathbb{R}^d$ is the first canonical unit vector. That is, assume the eigenvalue decomposition $B(q, e_1) = R(q, e_1)\Lambda(q, e_1)R(q, e_1)^{-1}$ is known. Hint: let Q be the rotation matrix that rotates e_1 to n, i.e., $n = Qe_1$ and use rotational invariance (do not need to explicitly construct Q). Note: This decomposition is crucial to construct Roe's approximate Riemann solver (and numerial flux) on general grids.