Design Optimization of Fluid Mechanical Systems using Reduced Order Models

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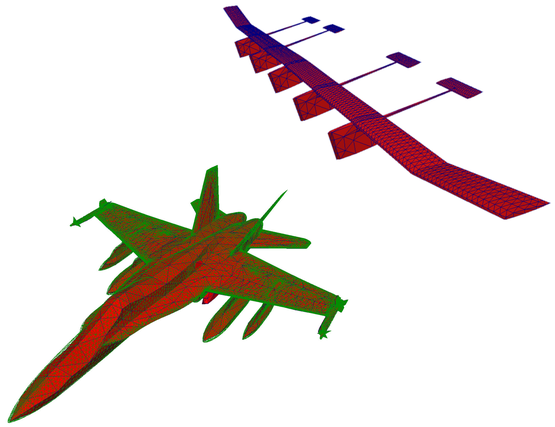
Motivation/Introduction

The incorporation of CFD-based optimization into the design of aircrafts would enable the automation of the iterative process that is currently driven by humans. This automation would enable the design of "better" aircrafts where the ordering of aircrafts is defined by some objective function. Unfortunately, full 3D CFD codes on industry-scale problems are too expensive to incorporate into an optimization loop, which may require thousands of iterations. This is where Reduced Order Models (ROMs) come into play. If we can construct a parametrically-robust ROM from only a few high-fidelity simulations and exploit this ROM throughout the design loop, there is potential for dramatic CPU savings.



A Reduced Order Model *IS*:

- a MODEL that is a low-dimensional approximation of some High-Dimensional Model (HDM)
- built on-top of a HDM

A Reduced Order Model *IS NOT*:

- a simplification of the HDM
- a response surface

PDE-Constrained Optimization

Let

$$R_s(q, z) = 0$$

denote the discretization of a steady PDE, where q is the state variable and z is the parameter or design variable. Also, let $\mathcal{L}(q, z)$ denote the quantity we would like to minimize. Then, the problem of interest is:

$$\begin{aligned} &\underset{q,z}{\text{minimize}} && \mathcal{L}(q, z) \\ &\text{subject to :} && R_s(q, z) = 0. \end{aligned}$$

- Nested Analysis and Design (NAND)
  - Use $R_s(q, z) = 0$ to write $q = q(z)$ and solve the unconstrained problem: $\underset{z}{\text{minimize}} \quad \mathcal{L}(q(z), z)$
- Simultaneous Analysis and Design (SAND)
  - Solve the original nonlinear programming problem over state and design variables.

Model Order Reduction

We are interested in the solution of the following system of ODEs

$$\frac{du}{dt} = F(u, z)$$

$$u \in \mathbb{R}^N - \text{state variable}$$

$$z \in \mathbb{R}^p - \text{design variable}$$

which may come from the semi-discretization of a PDE or represent the governing equations of a discrete system. Full discretization (with a 1-step scheme) yields:

$$R(u^{n+1}, z; u^n) = 0.$$

This is *Model I,* the HDM.

Now, we make the assumption that the solution lies in a low-dimensional affine subspace

$$u \approx \bar{u} + \Phi^{(i)} u_r, \qquad i \in \{1, 2, \ldots, N_v\}$$

$$u_r \in \mathbb{R}^{n_y} - \text{reduced coordinates.}$$

Substituting this into the discrete form, we reduced the number of unknowns resulting in an overdetermined system. To recover a square system of equations, we introduce the left bases $\Psi^{(i)}$ and force the equations to be orthogonal to its columns, yielding

$$\left(\Psi^{(i)}\right)^T R(\bar{u} + \Phi^{(i)} u_r^{n+1}, z; \bar{u} + \Phi^{(j)} u_r^n) = 0.$$

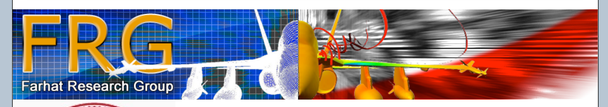
This is *Model II,* the ROM.

Right Basis Construction

- Let $\mathbb{X} \in \mathbb{R}^{N \times n_s}$ be a collection of solution snapshots from the HDM under different design parameter configurations, z_k for $k = 1, 2, \ldots, n$.
- Cluster columns of $\mathbb{X}$ into N_v disjoint sets $\mathbb{X}^{(1)}, \ldots, \mathbb{X}^{(N_v)}$, e.g. **kMeans**. This also defines cluster centers: $u_c^{(1)}, \ldots, u_c^{(N_v)}$.
- Reference the snapshots: $\mathbb{X}^{(i)} \leftarrow \mathbb{X}^{(i)} - \bar{u} e^T$.
- Use Proper Orthogonal Decomposition on each $\mathbb{X}^{(i)}$ to generate right bases $\Phi^{(i)}$.

Left Basis Choice

- Galerkin Approximation: $\Psi^{(i)} = \Phi^{(i)}$
  - "Optimal" for problems with SPD Jacobian
- Least-Square Petrov-Galerkin: $\Psi^{(i)} = \frac{\partial R}{\partial u}(u, z)\Phi^{(i)}$
  - "Optimal" for problems with non-SPD Jacobian



DOE CSGF

Online Exploitation of ROM (LSPG)

Suppose u^0 is the initial condition for the problem of interest, then the online ROM algorithm is:

- At step n, determine the cluster to which u^n is closest. Call this cluster i_n.
- Solve

$$\left(\Psi^{(i_n)}\right)^T R(\bar{u} + \Phi^{(i_n)} u_r^{n+1}, z; \bar{u} + \Phi^{(i_{n-1})} u_r^n) = 0$$

where

$$\Psi^{(i)} = \frac{\partial R}{\partial u}(u, z)\Phi^{(i)}$$

using the iteration

$$\begin{cases} u_r^{n+1,k} = u_r^n + \sum_{i=1}^{k-1} p_k \\ \left(\Phi^{(i_n)}\right)^T J_k^T J_k \Phi^{(i_n)} p_k = -\left(\Phi^{(i_n)}\right)^T J_k^T R_k \end{cases}$$

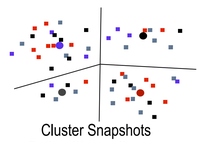
where

$$\begin{cases} R_k = R(\bar{u} + \Phi^{(i_n)} u_r^{n+1,k}, z) \\ J_k = \frac{\partial R}{\partial u}(\bar{u} + \Phi^{(i_n)} u_r^{n+1,k}, z). \end{cases}$$

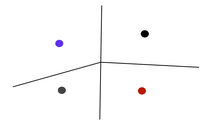
This is the globally-convergent Gauss-Newton method applied to

$$\underset{y \in \mathbb{R}^p}{\text{minimize}} \quad \left|\left| R(\bar{u} + \Phi^{(i)} y, z) \right|\right|_2.$$

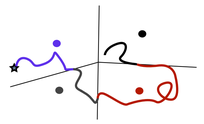
Collect Snapshots



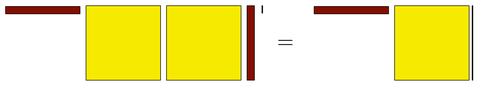
Cluster Snapshots
Define Cluster Centers



Discard Snapshots



Run ROM simulation;
Use "best" basis at each step



Linear System Schematic - LSPG

Hyperreduction

To speed up the bottleneck in the ROM (the formation of the linear system), we approximate the residual and the action of the Jacobian on the reduced order basis $\Phi^{(i)}$ as

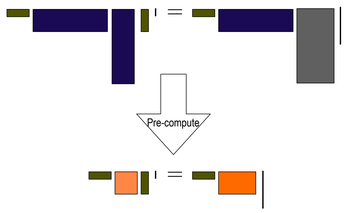
$$\begin{cases} R_k \approx \Phi_r^{(i)} R_{r,k} \\ J_k \Phi^{(i)} \approx \Phi_j^{(i)} J_{r,k} \end{cases}$$

where

$$R_{r,k} \in \mathbb{R}^{n_R}, \quad J_{r,k} \in \mathbb{R}^{n_J \times n_Y}$$

$$\Phi_r^{(i)} \in \mathbb{R}^{N \times n_R}, \quad \Phi_j^{(i)} \in \mathbb{R}^{N \times n_J}.$$

This is *Model III*. The method is called Gauss-Newton Approximated Tensors (GNAT).



Linear System Schematic - LSPG + Hyperreduction

Applications

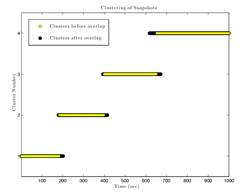
Burger's Equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0.02e^{0.02x}$$

$$u(t, 0) = \sqrt{5}$$

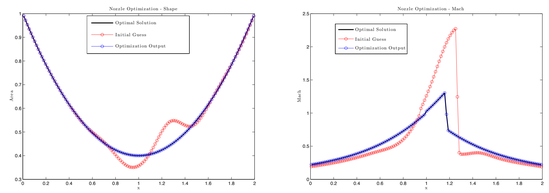
$$u(0, x) = 1$$

- HDM has 10,000 dof
- ROM uses 4 local bases of size 20, 14,10, and 4, respectively



Potential Nozzle Flow

- 1D Nozzle Problem with flow governed by potential equations
- Nozzle shape parametrized by cubic splines with 10 control points
- Parameter estimation
- NAND PDE-optimization framework



References

[1] David Amsallem, Matthew J. Zahr, and Charbel Farhat. Nonlinear model order reduction based on local reduced-order bases. *International Journal for Numerical Methods in Engineering*, 2012.

[2] K. Carlberg, C. Farhat, and C. Bou-Mosleh. Efficient Nonlinear Model Reduction via a Least-Squares Petrov-Galerkin Projection and Compressive Tensor Approximations. *International Journal for Numerical Methods in Engineering*, in press, May 2010.

[3] Michal Jerzy Rewienski. *A Trajectory Piecewise-Linear Approach to Model Order Reduction of Nonlinear Dynamical Systems*. PhD thesis, Massachusetts Institute of Technology, 2003.

[4] Kyle Washabaugh, David Amsallem, Matthew Zahr, and Charbel Farhat. Nonlinear model reduction for cfd problems using local reduced order bases. In *42nd AIAA Fluid Dynamics Conference and Exhibit*, New Orleans, LA, June 25-28 2012.