Unsteady PDE-Constrained Optimization using High-Order DG-FEM

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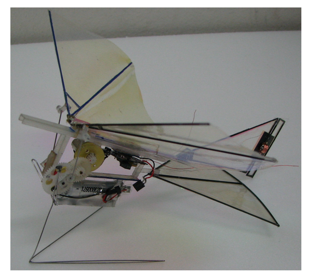
Motivation

- Design and control of physics-based systems commonly lead to optimization problems constrained by Partial Differential Equations (PDEs)
- Many of these problems are inherently unsteady:
  - Static systems without steady-state solutions (e.g. flow problems with separation and turbulence)
  - Dynamic systems (e.g. deforming domain problems such as flapping flight, wind turbines, etc)
- *Unsteady PDE-constrained optimization*



Introduction

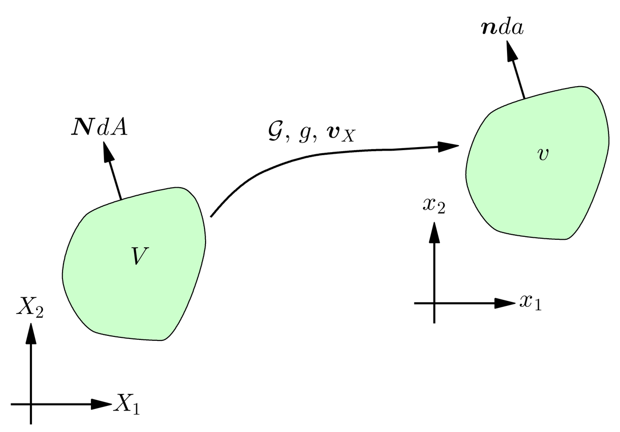
- Consider a general optimization problem involving a system of conservation laws on a deforming domain, with solution $\boldsymbol{U}$ and optimization parameters $\boldsymbol{\mu}$:

$$\begin{aligned} \underset{\boldsymbol{U},\,\boldsymbol{\mu}}{\text{minimize}} \quad & \int_{T_0}^{T_f} \int_{\boldsymbol{\Gamma}} f(\boldsymbol{U}, \boldsymbol{\mu}, t)\, dS\, dt \\ \text{subject to} \quad & \frac{\partial \boldsymbol{U}}{\partial t} + \nabla \cdot \boldsymbol{F}(\boldsymbol{U}, \nabla \boldsymbol{U}, \boldsymbol{\mu}) = 0 \end{aligned}$$

- Use high-order spatial and temporal discretizations and fully-discrete adjoint-based gradients to combat the high cost of unsteady optimization
- Application to flow-constrained trajectory optimization and energetically optimal flapping motions at constant thrust

High-Order Numerical Discretization

- Discretize the system of conservation laws domain using a high-order accurate Arbitrary Lagrangian-Eulerian (ALE) Discontinuous Galerkin Finite Element Method (DG-FEM)
- Introduce a time-dependent diffeomorphism $\mathcal{G}$ between a fixed reference domain V and the physical domain $v(t)$



- Transform state variables according to $\boldsymbol{U}_{\boldsymbol{X}} = g\boldsymbol{U}$, where $g = \det(\nabla_{\boldsymbol{X}} \mathcal{G})$, resulting in a modified system of conservation laws defined on the *reference* domain

$$\left.\frac{\partial \boldsymbol{U}_{\boldsymbol{X}}}{\partial t}\right|_{\boldsymbol{X}} + \nabla_{\boldsymbol{X}} \cdot \boldsymbol{F}_{\boldsymbol{X}}(\boldsymbol{U}_{\boldsymbol{X}},\, \nabla_{\boldsymbol{X}} \boldsymbol{U}_{\boldsymbol{X}}) = 0$$

- Discretize in space with DG-FEM to yield the semi-discrete system of equations

$$\mathbb{M} \frac{\partial \boldsymbol{u}}{\partial t} = \boldsymbol{r}(\boldsymbol{u}, \boldsymbol{\mu}, t)$$

- Discretize in time with a Diagonally Implicit Runge-Kutta (DIRK) scheme to obtain the fully-discrete equations

$$\begin{aligned} \boldsymbol{u}^{(0)} &= \boldsymbol{u}_0(\boldsymbol{\mu}) \\ \boldsymbol{u}^{(n)} &= \boldsymbol{u}^{(n-1)} + \sum_{i=1}^{s} b_i \boldsymbol{k}_i^{(n)} \\ \mathbb{M}\boldsymbol{k}_i^{(n)} &= \Delta t_n \boldsymbol{r}\left(\boldsymbol{u}_i^{(n)},\, \boldsymbol{\mu},\, t_i^{(n)}\right) \end{aligned}$$

- Discretize the output functional $J = \int_{T_0}^{T_f} \int_{\boldsymbol{\Gamma}} f(\boldsymbol{U}, \boldsymbol{\mu}, t)\, dS\, dt$ in a *solver-consistent* manner, i.e., the spatial integration uses the shape functions of the DG-ALE scheme and the temporal integration is performed with the same DIRK scheme

Fully-Discrete Adjoint Method

- The fully-discrete adjoint equations corresponding to the global numerical discretization of the equations are

$$\begin{aligned} \boldsymbol{\lambda}^{(N_t)} &= \frac{\partial J}{\partial \boldsymbol{u}^{(N_t)}}^T \\ \boldsymbol{\lambda}^{(n-1)} &= \boldsymbol{\lambda}^{(n)} + \frac{\partial J}{\partial \boldsymbol{u}^{(n-1)}}^T + \sum_{i=1}^{s} \Delta t_n \frac{\partial \boldsymbol{r}}{\partial \boldsymbol{u}}\left(\boldsymbol{u}_i^{(n)},\ \boldsymbol{\mu},\ t_i^{(n)}\right)^T \boldsymbol{\kappa}_i^{(n)} \\ \mathbb{M}^T \boldsymbol{\kappa}_i^{(n)} &= \sum_{j=i}^{s} a_{ji} \Delta t_n \frac{\partial \boldsymbol{r}}{\partial \boldsymbol{u}}\left(\boldsymbol{u}_j^{(n)},\ \boldsymbol{\mu},\ t_i^{(n)}\right)^T \boldsymbol{\kappa}_j^{(n)} \end{aligned}$$

- The adjoint states, $\boldsymbol{\lambda}^{(n)}$ and $\boldsymbol{\kappa}_i^{(n)}$, can be used to reconstruct the *gradient of an output functional with respect to parameters*, a crucial requirement for gradient-based optimization

$$\frac{\mathrm{d}J}{\mathrm{d}\boldsymbol{\mu}} = \frac{\partial J}{\partial \boldsymbol{\mu}} - \boldsymbol{\lambda}^{(0)^T} \frac{\partial \boldsymbol{u}_0}{\partial \boldsymbol{\mu}} - \sum_{n=1}^{N_t} \Delta t_n \sum_{i=1}^{s} \boldsymbol{\kappa}_i^{(n)^T} \frac{\partial \boldsymbol{r}}{\partial \boldsymbol{\mu}}(\boldsymbol{u}_i^{(n)},\ \boldsymbol{\mu},\ t_i^{(n)})$$

- Requires evolution of linear equations for each functional J whose derivative is desired. In the context of gradient-based optimization, this will be done for the **objective** function as well as all the **constraint** equations

Conclusions

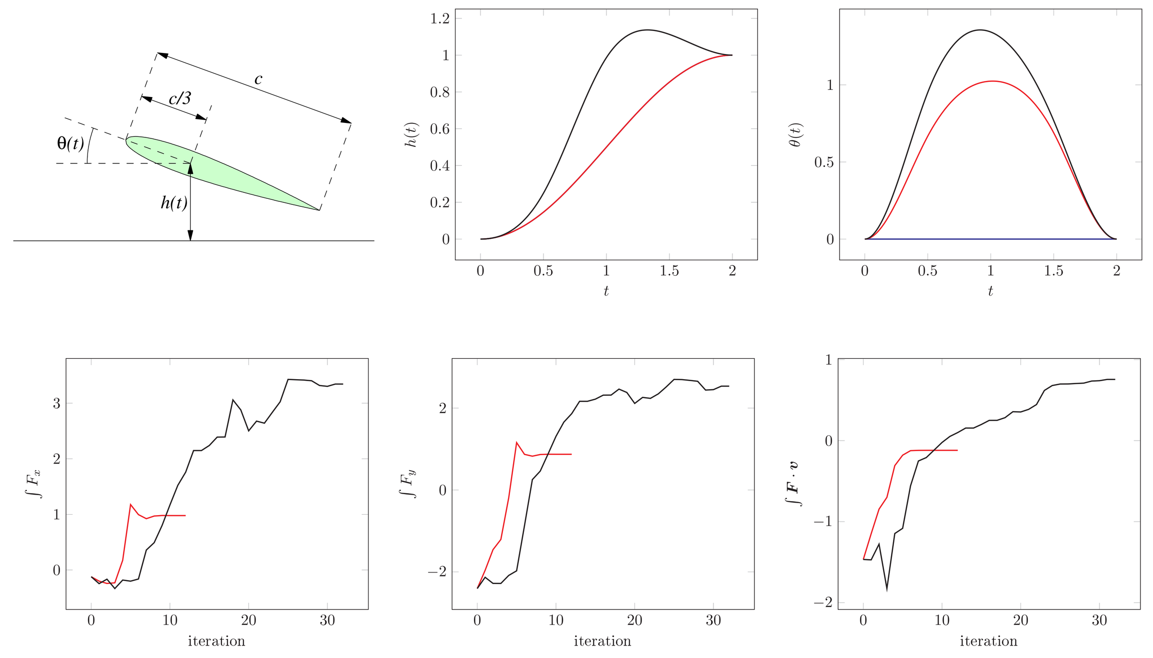
- A high-order DG-DIRK discretization of general conservation laws with a mapping-based ALE formulation for deforming domains
- A fully-discrete adjoint method for computing gradients of output functionals and constraints in optimization problems
- Framework demonstrated on the computation of energetically optimal motions of a 2D airfoil in a flow field with constraints

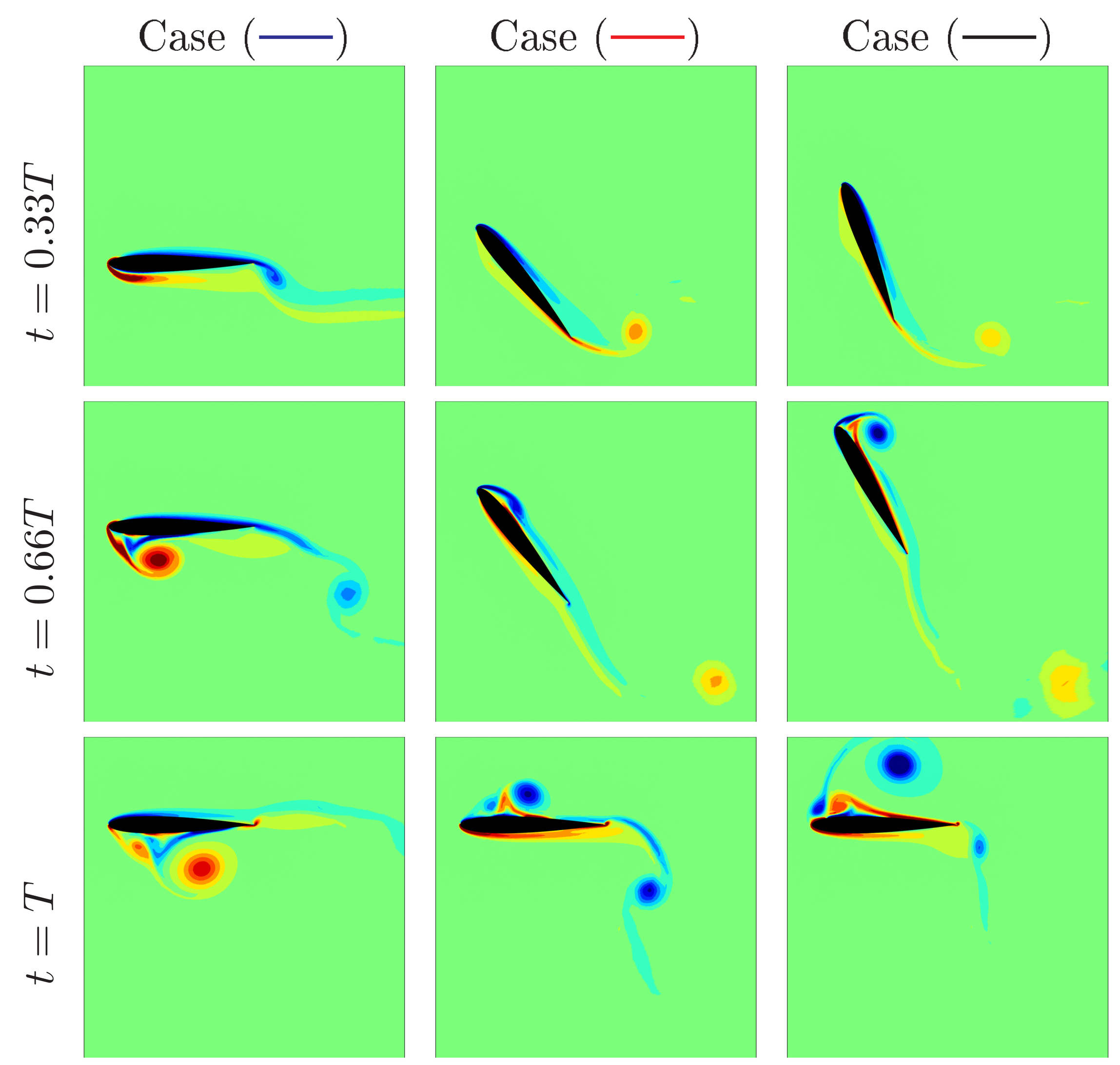
Energetically Optimal Trajectory

- Consider the trajectory optimization problem

$$\begin{aligned} \underset{h(t),\ \theta(t)}{\text{minimize}} \quad & \int_0^T \int_{\boldsymbol{\Gamma}} \boldsymbol{F} \cdot \boldsymbol{v}\, dS\, dt \\ \text{subject to} \quad & h(0) = h'(0) = h'(T) = 0,\ \ h(T) = 1 \\ & \theta(0) = \theta'(0) = \theta(T) = \theta'(T) = 0 \end{aligned}$$

- $h(t), \theta(t)$ discretized via *cubic splines* with 5 knots
- Optimization cases:
  - (blue) $h(t) = \frac{1}{2}(1 - \cos(\pi t/T)),\ \theta(t) = 0$
  - (red) $h(t) = \frac{1}{2}(1 - \cos(\pi t/T))$ fixed, $\theta(t)$ variable
  - (black) $h(t),\ \theta(t)$ variable





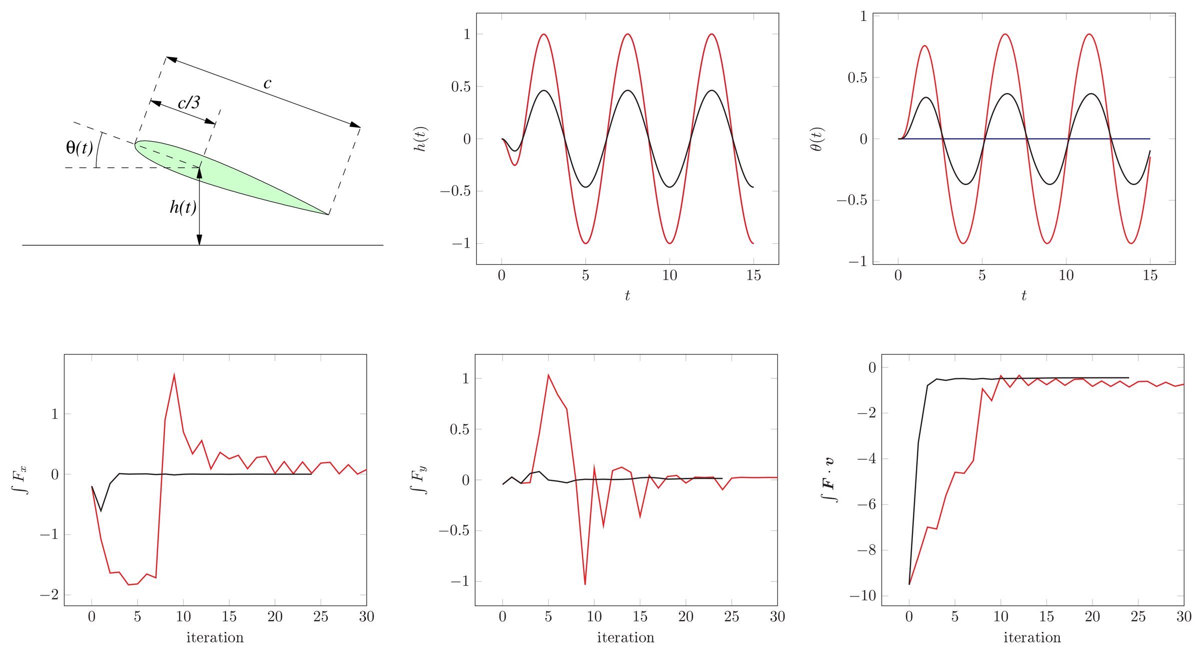
Vorticity field at three time instances for the three trajectories

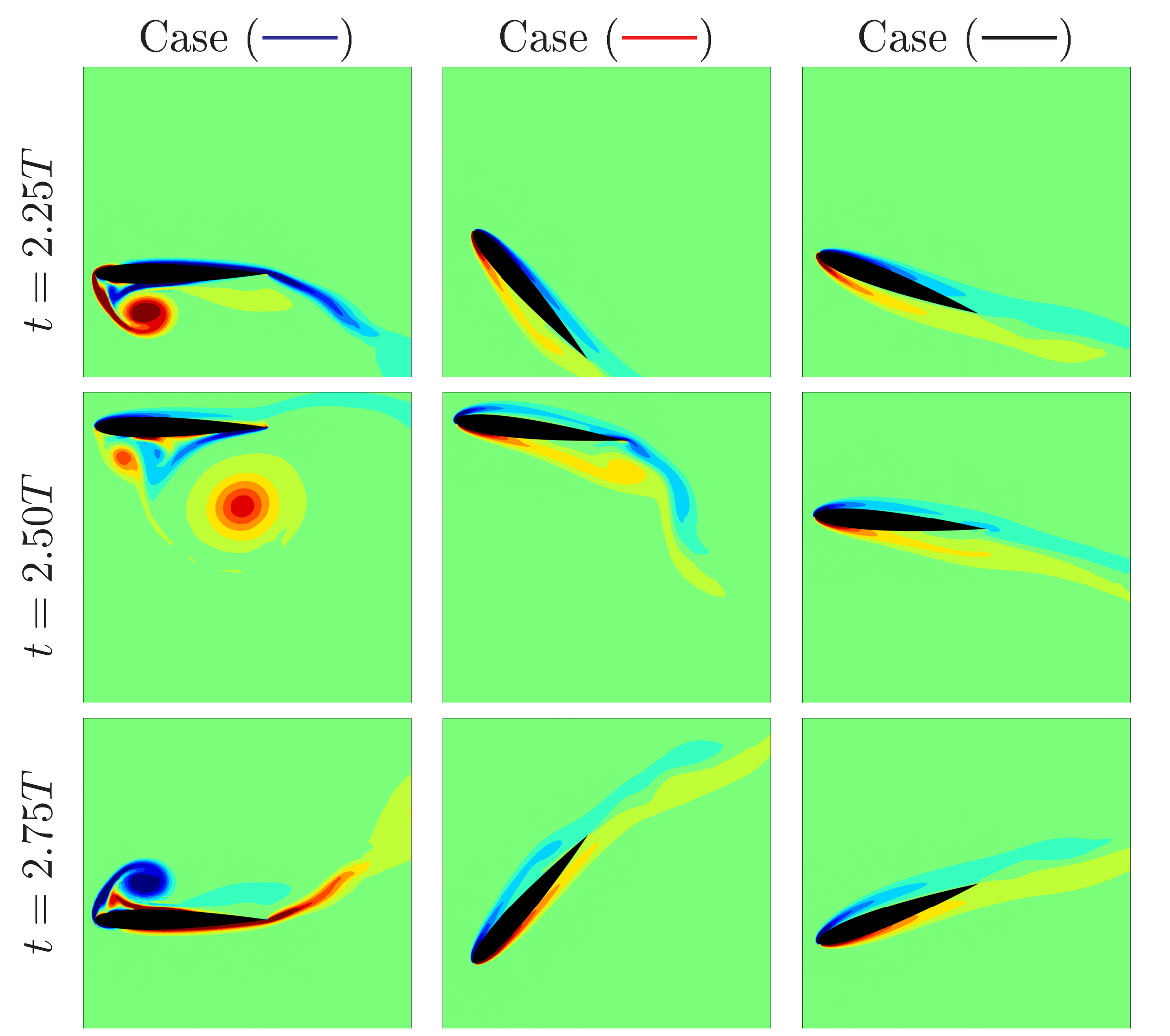
Energetically Optimal Flapping

- Consider the trajectory optimization problem

$$\begin{aligned} \underset{h(t),\ \theta(t)}{\text{minimize}} \quad & \int_{2T}^{3T} \int_{\boldsymbol{\Gamma}} \boldsymbol{F} \cdot \boldsymbol{v}\, dS\, dt \\ \text{subject to} \quad & \int_{2T}^{3T} \int_{\boldsymbol{\Gamma}} F_x\, dS\, dt = 0 \\ & h^{(k)}(t) = h^{(k)}(t + nT),\ \ \theta^{(k)}(t) = \theta^{(k)}(t + nT) \end{aligned}$$

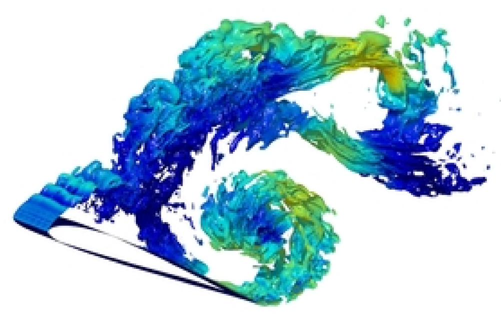
- $h(t), \theta(t)$ discretized via amplitude/phase of 5 *Fourier modes*
- Optimization cases:
  - (blue) $h(t) = -\cos(0.4\pi t/T),\ \theta(t) = 0$
  - (red) $h(t) = -\cos(0.4\pi t/T)$ fixed, $\theta(t)$ variable
  - (black) $h(t),\ \theta(t)$ variable





Vorticity field at three time instances for the three trajectories

Future Research



- Application of the method to real-world **3D problems**
- Extension of the method to **multiphysics** problems, such as FSI
- Extension of the method to **chaotic** problems, such as LES flows, where care must be taken to ensure the sensitivities are well-defined
- Incorporation of an **adaptive model reduction** technology to further reduce the cost of unsteady optimization