Efficient, Parametrically-Robust Nonlinear Model Reduction using Local Reduced-Order Bases

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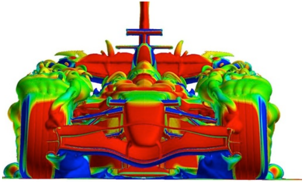
Motivation

- Complex, time-dependent problems



- Real-time analyses
  - Model Predictive Control
- Many-query analyses
  - Optimization
  - Uncertainty-Quantification

Model Order Reduction Framework

I. Full-order model → Data collection → Compression

↓ Approximation 1: Projection

II. Reduced-order model → Data collection → Compression

↓ Approximation 2: System approximation

III. Reduced-order model + system approximation

[Carlberg et. al. 2011]

High-Dimensional Model

Consider the nonlinear system of Ordinary Differential Equations (ODE), usually arising from the semi-discretization of Partial Differential Equation,

$$\frac{d\mathbf{w}}{dt} = \mathbf{F}(\mathbf{w}, t, \boldsymbol{\mu})$$

where

$\mathbf{w} \in \mathbb{R}^N$	state vector
$\boldsymbol{\mu} \in \mathbb{R}^d$	parameter vector
$\mathbf{F} : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^N$	nonlinearity of ODE

This is the High-Dimensional Model (HDM).

Fully Discretization of HDM

- Our approach to Model Order Reduction leverages dimensionality reduction at the **fully discrete** level
- Full, implicit (single-step) discretization of the governing equation yields a sequence of nonlinear systems of equations:

$$\mathbf{R}(\mathbf{w}^{(n)}, t_n, \boldsymbol{\mu}; \mathbf{w}^{(n-1)}) = 0, \qquad n \in \{1, 2, \ldots, N_s\}$$

where

$$\mathbf{w}^{(n)} = \mathbf{w}(t_n)$$
$$\mathbf{R} : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^N$$

From this point, we drop the dependence of $\mathbf{R}$ on the previous time step $\mathbf{w}^{(n-1)}$.

Model Order Reduction with Local Bases

- The goal of reducing the computational cost and resources required to solve a large-scale system of ODEs is attempted through **dimensionality reduction**
- Specifically, the (discrete) trajectory of the solution in state space is assumed to lie in a low-dimensional affine subspace

$$\mathbf{w}^{(n)} \approx \mathbf{w}^{(n-1)} + \Phi(\mathbf{w}^{(n-1)})\mathbf{y}^{(n)}$$

$\Phi(\mathbf{w}^{(n-1)}) \in \mathbb{R}^{N \times k_w(\mathbf{w}^{(n-1)})}$ Reduced Basis

$\mathbf{y}^{(n)} \in \mathbb{R}^{k_w(\mathbf{w}^{(n-1)})}$ Reduced Coordinates

where $k_w(\mathbf{w}^{(n-1)}) \ll N$

Overview

- In practice, N_V bases are computed in an offline phase: $\Phi^i \in \mathbb{R}^{N \times k_w^i}$
- Each basis, Φ^i, is associated with a representative vector in state space, $\mathbf{w}_c^i$
- Then, $\Phi(\mathbf{w}^{(n-1)}) \doteq \Phi^i$, where $||\mathbf{w}^{(n-1)} - \mathbf{w}_c^i|| \leq ||\mathbf{w}^{(n-1)} - \mathbf{w}_c^j||$ for all $j \in \{1, 2, \ldots, N_V\}$.

<u>Contrived Example</u>

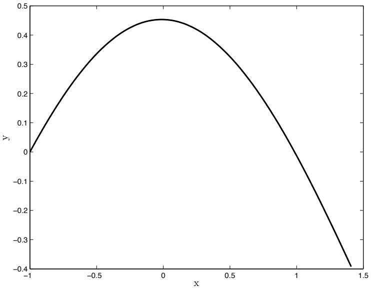
$$\frac{d}{dt}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{x(t)^2+y(t)^2} \\ -\frac{\sin x(t)}{x(t)^2+y(t)^2} \end{bmatrix}$$

$$\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

Data Collection

- HDM sampling (snapshot collection)
  - Simulate HDM at one or more parameter configurations $\{\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_n\}$ and collect snapshots $\mathbf{w}^{(j)}$
  - Combine in snapshot matrix $\mathbf{W}$

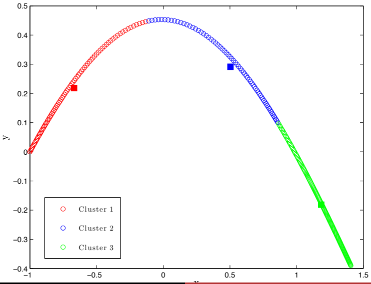
Figure : Contrived Example: HDM



Data Organization

- Snapshot clustering
  - Cluster snapshots using the k-means algorithm based on their relative distance in state space
  - Store the center of each cluster, $\mathbf{w}_c^i$
  - $\mathbf{W}$ partitioned into cluster snapshot matrices $\mathbf{W}_i$

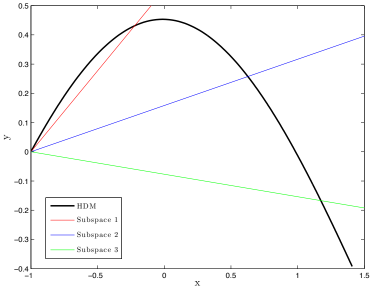
Figure : Contrived Example: Snapshot Clustering



Data Compression

- Modify snapshot matrices $\mathbf{W}_i$ by subtracting a reference vector, $\bar{\mathbf{w}}$ from each column $\hat{\mathbf{W}}_i = \mathbf{W}_i - \bar{\mathbf{w}}\mathbf{e}^T$
- Apply POD method to each cluster: $\Phi^i = \text{POD}(\hat{\mathbf{W}}_i)$

Figure : Contrived Example: Basis Construction



Overview

- The MOR assumption is substituted into the HDM to obtain the over-determined nonlinear system of equations:

$$\mathbf{R}(\mathbf{w}^{(n-1)} + \Phi^i \mathbf{y}^{(n)}, t_n, \boldsymbol{\mu}) = 0$$

- Since the above system does not have a solution, in general, we seek the solution that minimizes the residual of the HDM in the chosen affine subspace:

$$\mathbf{y}^{(n)} = \underset{\mathbf{y} \in \mathbb{R}^{k_w^i}}{\arg\min} ||\mathbf{R}(\mathbf{w}^{(n-1)} + \Phi^i \mathbf{y}, t_n, \boldsymbol{\mu})||_2$$

This is the Reduced-Order Model (ROM)

Inconsistency

- Recall the MOR assumption:

$$\mathbf{w}^{(n)} - \mathbf{w}^{(n-1)} \approx \Phi^i \mathbf{y}^{(n)}$$

$$\mathbf{w}^{(n)} - \mathbf{w}^{(switch)} \approx \Phi^i \sum_{k=switch}^{n} \mathbf{y}^{(k)}$$

where $\mathbf{w}^{(switch)}$ is the most recent state to initiate a switch between bases.

- Recall the reduced bases are constructed as

$$\Phi^i = \mathrm{POD}\left(\mathbf{W}_i - \bar{\mathbf{w}}\mathbf{e}^T\right)$$

- Basis construction consistent with MOR assumption only if $\bar{\mathbf{w}} = \mathbf{w}^{(switch)}$

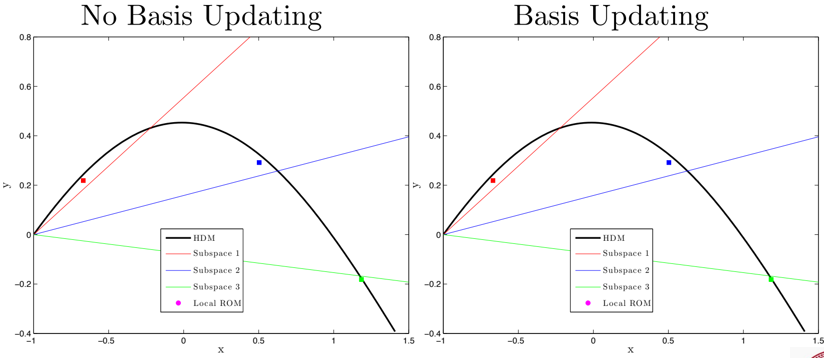
Solution: Fast Basis Updating

- We seek a reduced basis of the form:

$$\begin{aligned}\hat{\Phi}_i &= POD(\mathbf{W}_i - \mathbf{w}^{(switch)}\mathbf{e}^T) \\ &= POD(\mathbf{W}_i - \bar{\mathbf{w}}\mathbf{e}^T + (\bar{\mathbf{w}} - \mathbf{w}^{(switch)})\mathbf{e}^T) \\ &= POD(\hat{\mathbf{W}}_i + (\bar{\mathbf{w}} - \mathbf{w}^{(switch)})\mathbf{e}^T)\end{aligned}$$

- $\hat{\Phi}$ is the (truncated) left singular vectors of a matrix that is a rank-one update of a matrix, $\hat{\mathbf{W}}_i$, whose (truncated) left singular vectors is readily available, Φ_i.
- Fast updates available [Brand 2006].

Figure : Contrived Example: ROM Solution



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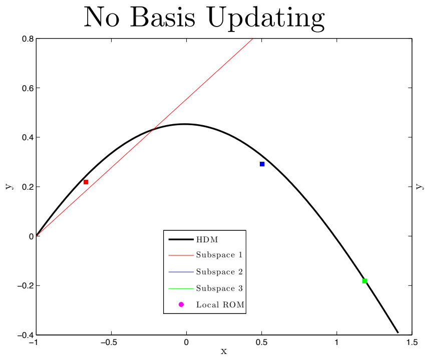

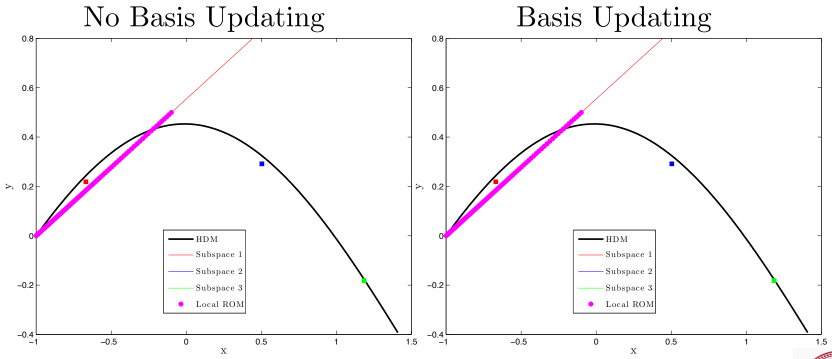

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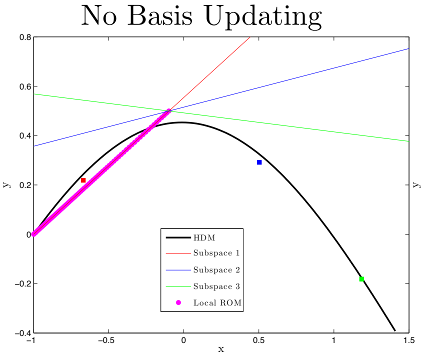

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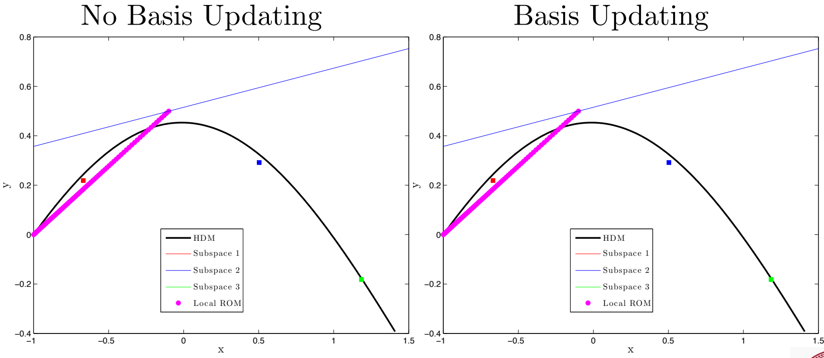

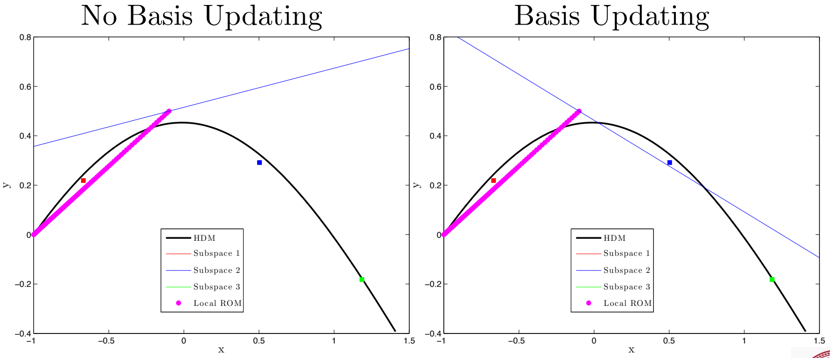

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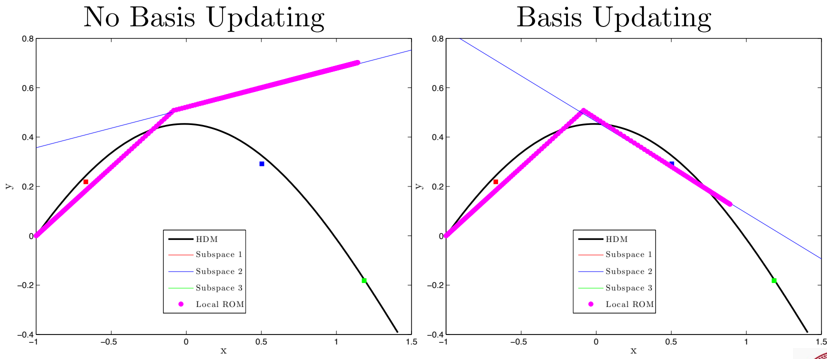

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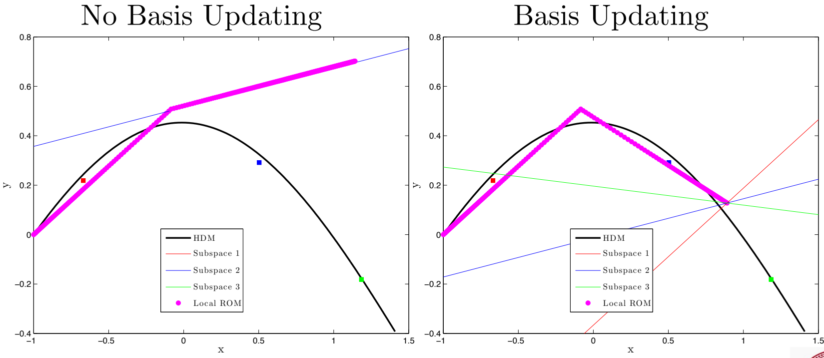

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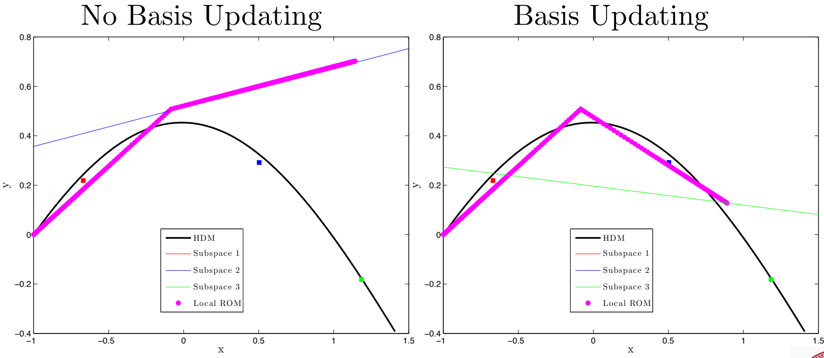

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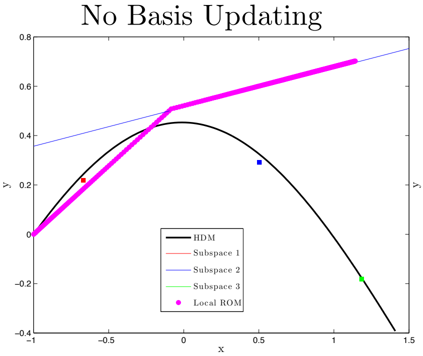

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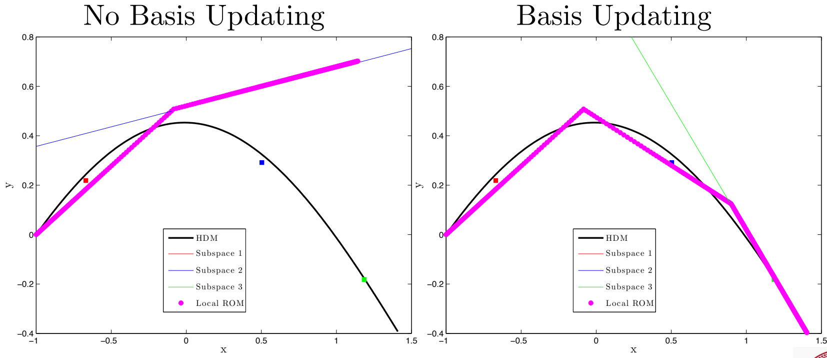

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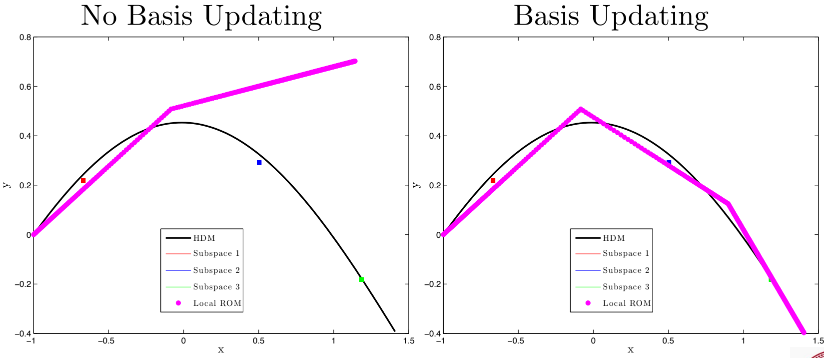


Extension to Hyperreduction (hROM)

- For many classes of ODEs, the above framework is not sufficient to achieve speedups or a reduction in required computational resources
  - e.g. nonlinear, time-variant, or parametric ODEs
- For the nonlinear case, methods exist for creating reduced bases Φ_R^i and Φ_J^i for the nonlinear residual and Jacobian, respectively [Chaturantabut and Sorensen 2009, Carlberg et al 2011].
  - Enables pre-computation of terms that were previously iteration-dependent
- Further reduction available by using a *sample mesh*, i.e. a well-chosen subset of the entire mesh [Carlberg et. al. 2011].

1D Burger's Equation (Shock Propagation)

High-Dimensional Model

- $N = 10,000$ degrees of freedom

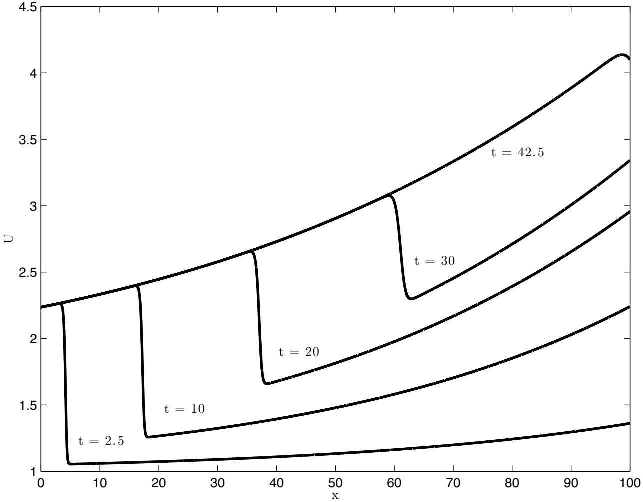
$$\frac{\partial U(x,t)}{\partial t} + \frac{\partial f(U(x,t))}{\partial x} = g(x) \quad \forall x \in [0, L]$$
$$U(x,0) = 1, \quad \forall x \in [0, L]$$
$$U(0,t) = u(t), \quad t > 0$$

where $g(x) = 0.02e^{0.02x}$, $f(U) = 0.5U^2$, and $u(t) = 5$.

Reduced-Order Model

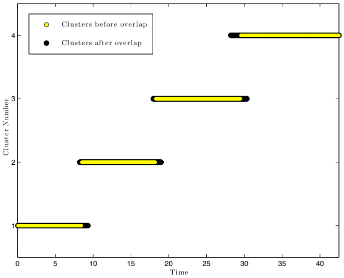
- $N_V = 4$ bases of size: 9, 5, 4, 4

High-Dimensional Model

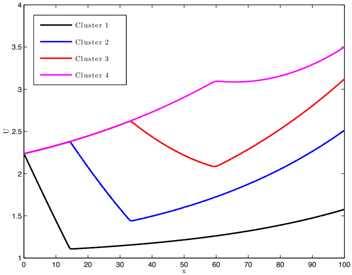


Clustering Results

Snapshot Clustering

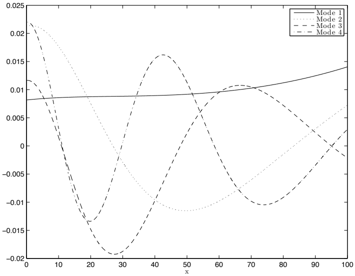


Cluster Centers

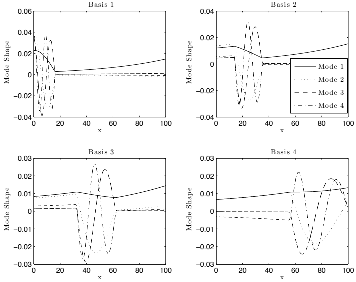


Reduced Basis Modes

Global Basis

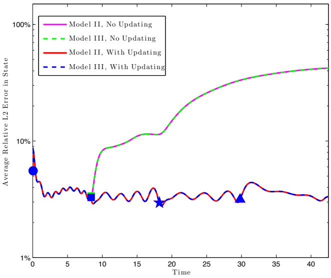


Local Bases

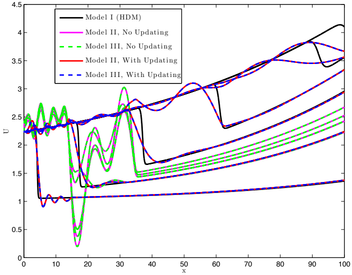


Simulation Results

Error vs. Time

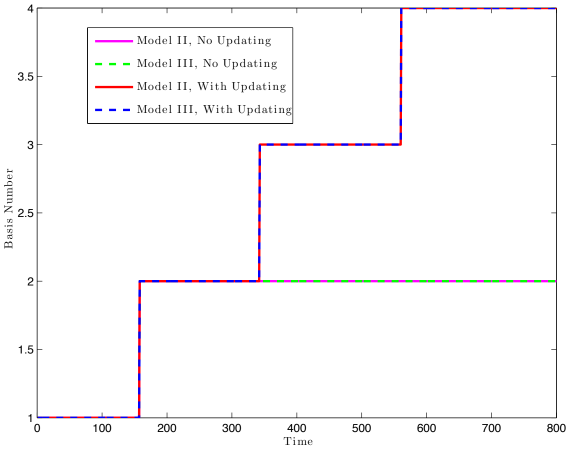


Solution Snapshots



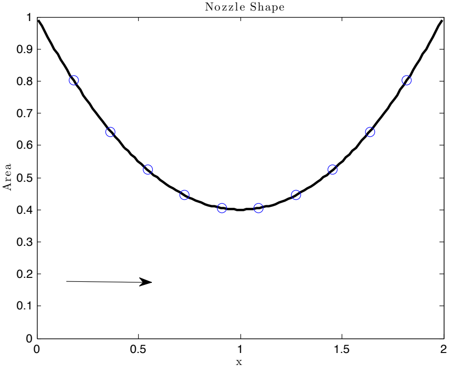
Symbols indicate basis switch

Basis Usage


Model II, No Updating
Model III, No Updating
Model II, With Updating
Model III, With Updating
Basis Number
Time
1
1.5
2
2.5
3
3.5
4
0
100
200
300
400
500
600
700
800

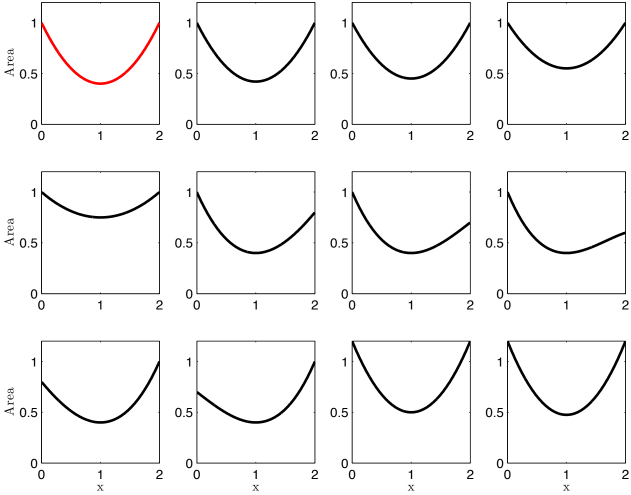
Potential Nozzle Flow

$$\frac{d}{dx}\left(A(x)\rho(x)u(x)\right) = 0 \tag{1}$$

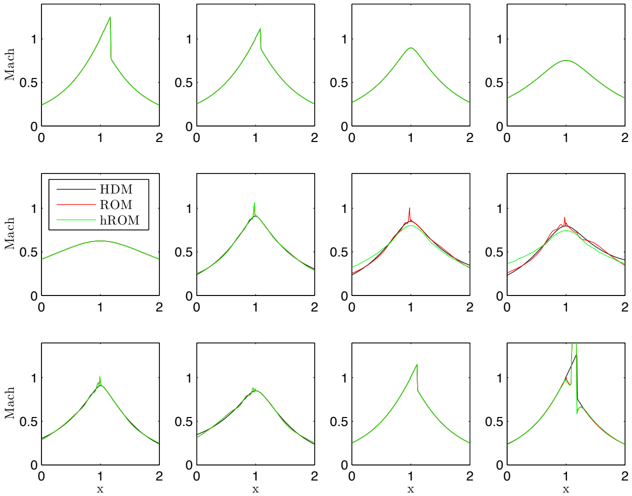


Parametric Study - Setup

Training Online

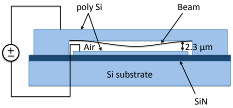


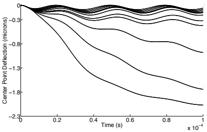
Parametric Study - Results



Other Application: MEMS

- Parametric study





Model	Degrees of freedom	GNAT	Relative error	CPU time (s)	speedup
HDM	N = 4050	-	-	317	-
ROM with exact update	k = (8,8)	k_r = (20,20) $\mathcal{I}$ = (20,20)	0.57%	18.24	17.37
ROM with approximate update $n_Q = 1$	k = (8,8)	k_r = (20,20) $\mathcal{I}$ = (20,20)	0.28%	17.34	18.28

Conclusions

- Local model reduction method
  - attractive for problems with distinct solution regimes
  - model reduction assumption and data collection are inconsistent
- Local model reduction with online basis updates
  - addresses inconsistency of local MOR
  - injects “online” data into pre-computed basis
- Future work
  - application to 3D turbulent flows
  - application to nonlinear structural dynamics
  - use as surrogate in PDE-constrained optimization and uncertainty quantification
- References

  - Amsallem, D., Zahr, M. J., and Farhat, C., “Nonlinear Model Order Reduction Based on Local ReducedOrder Bases,” International Journal for Numerical Methods in Engineering, 2012.
  - Washabaugh, K., Amsallem, D., Zahr, M., and Farhat, C., “Nonlinear Model Reduction for CFD Problems Using Local Reduced Order Bases,” 42nd AIAA Fluid Dynamics Conference and Exhibit, New Orleans, LA, June 25-28 2012.

Acknowledgements

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