Rapid Topology Optimization using Reduced-Order Models

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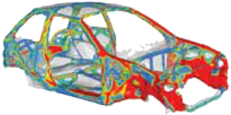
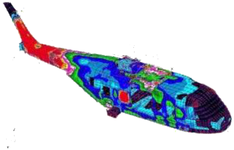
12th U.S. National Congress on Computational Mechanics
Raleigh, North Carolina
July 22 - 25, 2013

Motivation

- For industry-scale design problems, topology optimization is a beneficial tool that is *time and resource intensive*
    - Large number of calls to structural solver usually required
    - Each structural call is expensive, especially for nonlinear 3D High-Dimensional Models (HDM)
- Use a Reduced-Order Model (ROM) as a surrogate for the structural model in a material topology optimization loop
    - Large speedups over HDM realized





0-1 Material Topology Optimization

$$\begin{aligned} \underset{\boldsymbol{\chi} \in \mathbb{R}^{n_{el}}}{\text{minimize}} \quad & \mathcal{L}(\mathbf{u}(\boldsymbol{\chi}), \boldsymbol{\chi}) \\ \text{subject to} \quad & \mathbf{c}(\mathbf{u}(\boldsymbol{\chi}), \boldsymbol{\chi}) \leq 0 \end{aligned}$$

- $\mathbf{u}$ is implicitly defined as a function of $\boldsymbol{\chi}$ through the HDM equation

$$\mathbf{f}^{int}(\mathbf{u}) = \mathbf{f}^{ext}$$

$$\mathbb{C}^e = \mathbb{C}_0^e \boldsymbol{\chi}_e \qquad \rho^e = \rho_0^e \boldsymbol{\chi}_e \qquad \boldsymbol{\chi}_e = \begin{cases} 0, & e \notin \Omega^* \\ 1, & e \in \Omega^* \end{cases}$$

- Assume geometric *nonlinearity* and linear material law
  - Large deformations of St. Venant-Kirchhoff material

Reduced-Order Model

- Model Order Reduction (MOR) assumption
  - State vector lies in low-dimensional subspace defined by a Reduced-Order Basis (ROB) $\mathbf{\Phi} \in \mathbb{R}^{N \times k_{\mathbf{u}}}$

$$\mathbf{u} \approx \mathbf{\Phi y}$$

  - $k_{\mathbf{u}} \ll N$
- N equations, $k_{\mathbf{u}}$ unknowns

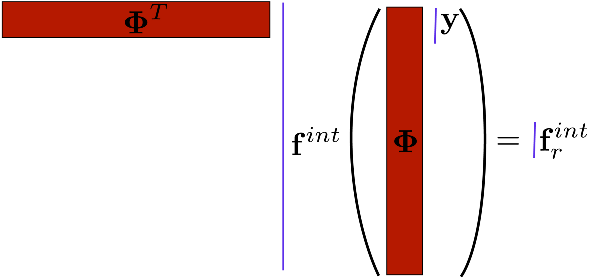
$$\mathbf{f}^{int}(\mathbf{\Phi y}) = \mathbf{f}^{ext}$$

- Galerkin projection

$$\mathbf{\Phi}^T \mathbf{f}^{int}(\mathbf{\Phi y}) = \mathbf{\Phi}^T \mathbf{f}^{ext}$$

NL ROM Bottleneck - Internal Force

$$\boldsymbol{\Phi}^T \mathbf{f}^{int}(\boldsymbol{\Phi}\mathbf{y}) = \boldsymbol{\Phi}^T \mathbf{f}^{ext}$$



NL ROM Bottleneck - Tangent Stiffness

$$\boldsymbol{\Phi}^T \mathbf{f}^{int}(\boldsymbol{\Phi}\mathbf{y}) = \boldsymbol{\Phi}^T \mathbf{f}^{ext}$$

$$\boldsymbol{\Phi}^T \quad \frac{\partial \mathbf{f}^{int}}{\partial \mathbf{u}}(\boldsymbol{\Phi}\mathbf{y}) \quad \boldsymbol{\Phi} = \mathbf{K}_r$$

Internal Force

The expression for the internal force is

$$\mathbf{f}_{jL}^{int} = \int_{\Omega_0} \mathbf{P}_{ij} \frac{\partial \mathbf{N}_L}{\partial \mathbf{X}_i} \, d\mathbf{X}$$

where $\mathbf{N}_I(\mathbf{X})$ is the shape function corresponding to node I and

$$\begin{aligned}
\mathbf{u}_i(\mathbf{X}) &= \mathbf{u}_{iI}\mathbf{N}_I(\mathbf{X}) && \text{(FEM discretization)} \\
\mathbf{F} &= \mathbf{I} + \mathbf{u}\frac{\partial \mathbf{N}}{\partial \mathbf{X}} && \text{(Deformation Gradient)} \\
\mathbf{E} &= \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I}) && \text{(Green-Lagrange Strain)} \\
\mathbf{P} &= \mathbf{S}\mathbf{F}^T && \text{(First Piola-Kirchhoff Stress)} \\
\mathbf{S} &= \lambda(\mathbf{X})\text{tr }(\mathbf{E})\mathbf{I} + 2\mu(\mathbf{X})\mathbf{E} && \text{(Second Piola-Kirchhoff Stress)}
\end{aligned}$$

Internal Force - Cubic Polynomial in Displacements

$$
\begin{aligned}
\mathbf{f}_{jL}^{int} &= \int_{\Omega_0} \mathbf{P}_{ij} \frac{\partial \mathbf{N}_I}{\partial \mathbf{X}_i} \, d\mathbf{X} \\
&= \bar{\mathbf{A}}_{jtIL} \mathbf{u}_{tI} + \bar{\mathbf{B}}_{LI} \mathbf{u}_{jI} + \\
\bar{\mathbf{C}}_{LIJj} \mathbf{u}_{kI} \mathbf{u}_{kJ} &+ \widehat{\mathbf{C}}_{ILQt} \mathbf{u}_{jQ} \mathbf{u}_{tI} + \bar{\mathbf{D}}_{IJQL} \mathbf{u}_{kI} \mathbf{u}_{kJ} \mathbf{u}_{jQ}
\end{aligned}
$$

where,

$$
\begin{aligned}
\bar{\mathbf{A}} &= \bar{\mathbf{A}}\left(\Omega, \lambda(\mathbf{X})\right) \\
\bar{\mathbf{B}} &= \bar{\mathbf{B}}(\Omega, \mu(\mathbf{X})) \\
\bar{\mathbf{C}} &= \bar{\mathbf{C}}(\Omega, \lambda(\mathbf{X}), \mu(\mathbf{X})) \\
\widehat{\mathbf{C}} &= \widehat{\mathbf{C}}(\Omega, \lambda(\mathbf{X}), \mu(\mathbf{X})) \\
\bar{\mathbf{D}} &= \bar{\mathbf{D}}(\Omega, \lambda(\mathbf{X}), \mu(\mathbf{X}))
\end{aligned}
$$

Material Representation

Let material distributions be represented with the basis functions:

$$\begin{aligned} \lambda(\mathbf{X}) &= \phi_i^{\lambda}(\mathbf{X})\alpha_i^r, & i &= 1, 2, \ldots, n_\alpha \\ \mu(\mathbf{X}) &= \phi_i^{\mu}(\mathbf{X})\alpha_i^r, & i &= 1, 2, \ldots, n_\alpha \\ \rho(\mathbf{X}) &= \phi_i^{\rho}(\mathbf{X})\alpha_i^r, & i &= 1, 2, \ldots, n_\alpha. \end{aligned}$$

Then

$$\begin{aligned} \bar{\mathbf{A}} &= \bar{\mathbf{A}}\left(\Omega, \phi_i^{\lambda}\right)\alpha_i^r \\ \bar{\mathbf{B}} &= \bar{\mathbf{B}}(\Omega, \phi_i^{\mu})\alpha_i^r \\ \bar{\mathbf{C}} &= \bar{\mathbf{C}}(\Omega, \phi_i^{\lambda}, \phi_i^{\mu})\alpha_i^r \\ \widehat{\mathbf{C}} &= \widehat{\mathbf{C}}(\Omega, \phi_i^{\lambda}, \phi_i^{\mu})\alpha_i^r \\ \bar{\mathbf{D}} &= \bar{\mathbf{D}}(\Omega, \phi_i^{\lambda}, \phi_i^{\mu})\alpha_i^r \end{aligned}$$

Pre-computed ROM - cubic nonlinearity

- HDM

$$\mathbf{f}^{int}_{jL} = \bar{\mathbf{A}}_{jtIL}\mathbf{u}_{tI} + \bar{\mathbf{B}}_{LI}\mathbf{u}_{jI} + \bar{\mathbf{C}}_{LIJj}\mathbf{u}_{kI}\mathbf{u}_{kJ} + \widehat{\mathbf{C}}_{ILQt}\mathbf{u}_{jQ}\mathbf{u}_{tI} + \bar{\mathbf{D}}_{IJQL}\mathbf{u}_{kI}\mathbf{u}_{kJ}\mathbf{u}_{jQ}$$

- ROM

$$\left[\mathbf{\Phi}^T\mathbf{f}^{int}(\mathbf{\Phi}\mathbf{y})\right]_r = \boldsymbol{\beta}_{rp}\mathbf{y}_p + \boldsymbol{\gamma}_{rpq}\mathbf{y}_p\mathbf{y}_q + \boldsymbol{\omega}_{rpqt}\mathbf{y}_p\mathbf{y}_q\mathbf{y}_t$$

$$\boldsymbol{\beta} = \boldsymbol{\beta}(\mathbf{\Phi}, \phi_i^{\lambda}, \phi_i^{\mu})\alpha_i^r$$
$$\boldsymbol{\gamma} = \boldsymbol{\gamma}(\mathbf{\Phi}, \phi_i^{\lambda}, \phi_i^{\mu})\alpha_i^r$$
$$\boldsymbol{\omega} = \boldsymbol{\omega}(\mathbf{\Phi}, \phi_i^{\lambda}, \phi_i^{\mu})\alpha_i^r$$

$$\boxed{\mathbf{\Phi}^T\mathbf{f}^{int}(\mathbf{\Phi}\mathbf{y}) = \mathbf{\Phi}^T\mathbf{f}^{ext}}$$

ROM Pre-computation Approach

Advantages

- Only need to solve small, cubic nonlinear system online
- Large speedups possible without hyperreduction, $\mathcal{O}(10^3)$
- Amenable to 0-1 material topology optimization
  - $\boldsymbol{\alpha}^r$ provide control over material distribution
  - $\boldsymbol{\alpha}^r$ can be used as optimization variables

Disadvantages

- *Currently* limited to StVK material, *Lagrangian* elements
- Offline cost scales as $\mathcal{O}(n_\alpha \cdot n_{el} \cdot k_{\mathbf{u}}^4)$
- Offline storage scales as $\mathcal{O}(n_\alpha \cdot k_{\mathbf{u}}^4)$
- Online storage scales as $\mathcal{O}(k_{\mathbf{u}}^4)$
- Can only vary material distribution in the subspace defined by the material snapshot vectors

Reduced Topology Optimization

$$\begin{aligned} \underset{\boldsymbol{\alpha}_r \in \mathbb{R}^{n_\alpha}}{\text{minimize}} \quad & \mathcal{L}(\mathbf{y}(\boldsymbol{\alpha}_r), \boldsymbol{\alpha}_r) \\ \text{subject to} \quad & \mathbf{c}(\mathbf{y}(\boldsymbol{\alpha}_r), \boldsymbol{\alpha}_r) \leq 0 \end{aligned}$$

- $\mathbf{y}$ is implicitly defined as a function of $\boldsymbol{\alpha}_r$ through the ROM equation

$$\boldsymbol{\Phi}^T \mathbf{f}^{int}(\boldsymbol{\Phi}\mathbf{y}) = \boldsymbol{\Phi}^T \mathbf{f}^{ext}$$

Structural Simulation

- St. Venant-Kirchhoff
- 66,191 tetrahedral elements
- 13,110 nodes, 38,664 dof
- Static simulation with load applied in 10 increments
- Loads: Bending, Twisting, Self-Weight
- ROM size: $k_{\mathbf{u}} = 5$

Simulation Results

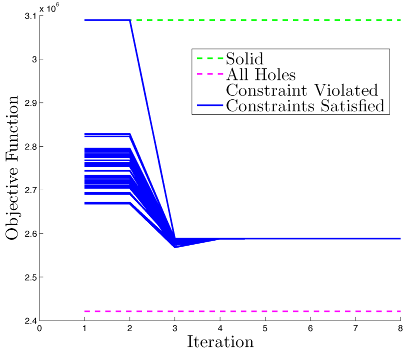
	Offline (s)	Online (s)	Speedup	Error (%)
HDM	-	750	-	-
ROM	0.38	170	3.96	0.003
ROM-precomp	5,171	0.37	2,051	0.003

Optimization Setup

- Minimize structural weight
- Constraint on maximum vertical displacement
- 46 Material Snapshots
  - 45 possible voids
  - volume surrounding all possible voids

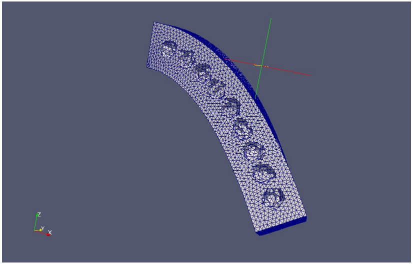
Material Snapshots

Optimization Results



Optimization Iterates
(Location of Voids)

Optimization Results

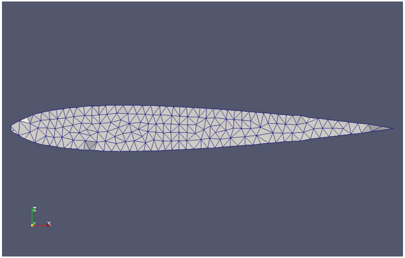


Deformed Configuration (Optimal Solution)

	Initial Guess	Optimal Solution
Structural Weight	2.776×10^6	2.588×10^6
Constraint Violation	9.96×10^{-2}	1.34×10^{-10}

Problem Setup

- St. Venant-Kirchhoff
- 90,799 tetrahedral elements
- 29,252 nodes, 86,493 dof
- Static simulation with load applied in 10 increments
- Loads: Bending (X- and Y- axis), Twisting, Self-Weight
- ROM size: $k_{\mathbf{u}} = 5$



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Problem Setup

- St. Venant-Kirchhoff
- 90,799 tetrahedral elements
- 29,252 nodes, 86,493 dof
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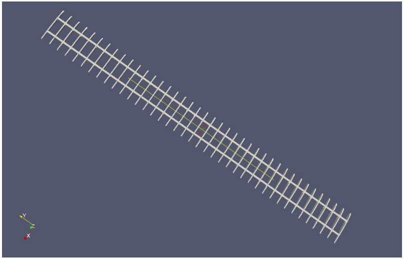


40 Ribs

Problem Setup

- St. Venant-Kirchhoff
- 90,799 tetrahedral elements
- 29,252 nodes, 86,493 dof
- Static simulation with load applied in 10 increments
- Loads: Bending (X- and Y- axis), Twisting, Self-Weight
- ROM size: $k_{\mathbf{u}} = 5$

Simulation Results

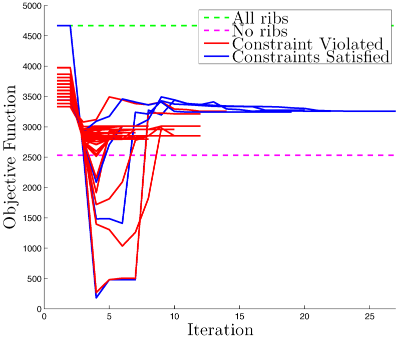
	Offline (s)	Online (s)	Speedup	Error (%)
HDM	-	811	-	-
ROM	1.01	376	2.16	0.002
ROM-precomp	9,603	1.51	538	1.73

Optimization Setup

- Minimize structural weight
- Constraint on maximum vertical horizontal displacements
- 41 Material Snapshots
  - 40 possible ribs
  - two spars jointly

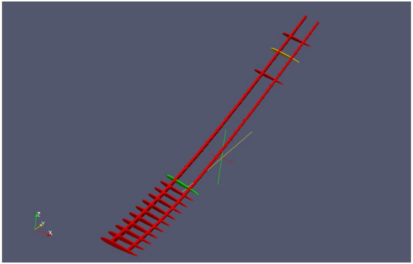
Material Snapshots

Optimization Results



Optimization Iterates

Optimization Results



Deformed Configuration (Optimal Solution)

	Initial Guess	Optimal Solution
Structural Weight	3.44×10^{3}	3.24×10^{3}
Constraint Violation	4.85×10^{-2}	1.19×10^{-16}

Conclusion and Future Work

- New method for material topology optimization using reduced-order models
  - $\mathcal{O}(10^3)$ speedup over HDM
- Strongly enforce manufacturability constraints
  - selection of material snapshots and optimization constraints
- Potential to address large problems
- Investigate extending method to more sophisticated topology optimization techniques