Robust Reduced-Order Models Via Fast, Low-Rank Basis Updates

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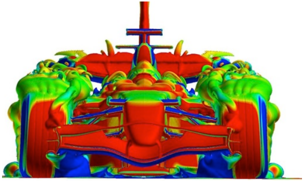
CP1: Reduced Order Models

Motivation

- Complex, time-dependent problems



- Real-time analyses
  - Model Predictive Control
- Many-query analyses
  - Optimization
  - Uncertainty Quantification

High-Dimensional Model

Consider the sequence of nonlinear system of equations, usually arising from the discretization of PDE,

$$\mathbf{R}^{(n)}(\mathbf{w}^{(n)}, t_n, \boldsymbol{\mu}) = 0$$

where

$\mathbf{w} \in \mathbb{R}^N$	state vector
$\boldsymbol{\mu} \in \mathbb{R}^d$	parameter vector
$\mathbf{R}^{(n)} : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^N$	governing equations

This is the High-Dimensional Model (HDM).

Model Order Reduction with Local Bases

- The goal of reducing the computational cost and resources required to solve a large-scale system of ODEs is attempted through **dimensionality reduction**
- Specifically, the (discrete) trajectory of the solution in state space is assumed to lie in a low-dimensional affine subspace

$$\mathbf{w}^{(n)} \approx \mathbf{w}^{(n-1)} + \boldsymbol{\Phi}(\mathbf{w}^{(n-1)})\mathbf{y}^{(n)}$$

$\boldsymbol{\Phi}(\mathbf{w}^{(n-1)}) \in \mathbb{R}^{N \times k_w(\mathbf{w}^{(n-1)})}$ Reduced Basis

$\mathbf{y}^{(n)} \in \mathbb{R}^{k_w(\mathbf{w}^{(n-1)})}$ Reduced Coordinates

where $k_w(\mathbf{w}^{(n-1)}) \ll N$ [Amsallem, Zahr, Farhat 2012]

$$\boxed{\boldsymbol{\Psi}(\mathbf{w}^{(n-1)})^T \mathbf{R}^{(n)}(\mathbf{w}^{(n-1)} + \boldsymbol{\Phi}(\mathbf{w}^{(n-1)})\mathbf{y}^{(n)}) = 0}$$

Contrived Example

The details of the local ROM framework will be exampled in the context of a *contrived* example:

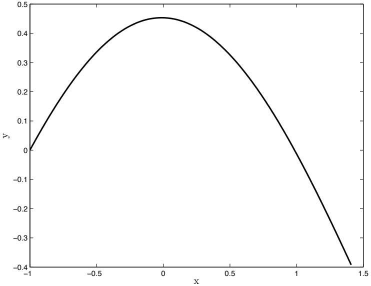
$$\frac{d}{dt}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{x(t)^2+y(t)^2} \\ -\frac{\sin x(t)}{x(t)^2+y(t)^2} \end{bmatrix}$$

$$\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

Data Collection

- HDM sampling (snapshot collection)
  - Simulate HDM at one or more parameter configurations $\{\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_n\}$ and collect snapshots $\mathbf{w}^{(j)}$
  - Combine in snapshot matrix $\mathbf{W}$

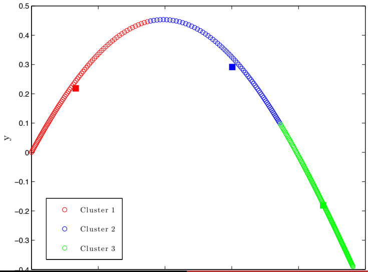
Figure : Contrived Example: HDM



Data Organization

- Snapshot clustering
  - Cluster snapshots using the k-means algorithm based on their relative distance in state space
  - Store the center of each cluster, $\mathbf{w}_c^i$
  - $\mathbf{W}$ partitioned into cluster snapshot matrices $\mathbf{W}$

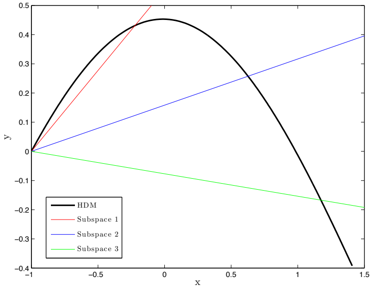
Figure : Contrived Example: Snapshot Clustering



Data Compression

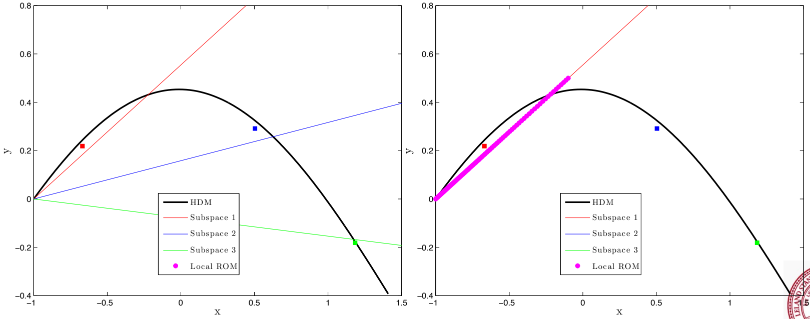
- Modify snapshot matrices $\mathbf{W}_i$ by subtracting a reference vector, $\bar{\mathbf{w}}$ from each column $\hat{\mathbf{W}}_i = \mathbf{W}_i - \bar{\mathbf{w}}\mathbf{e}^T$
    - usually the mean or initial condition
- Apply POD method to each cluster: $\mathbf{\Phi}^i = \text{POD}(\hat{\mathbf{W}}_i)$

Figure : Contrived Example: Basis Construction



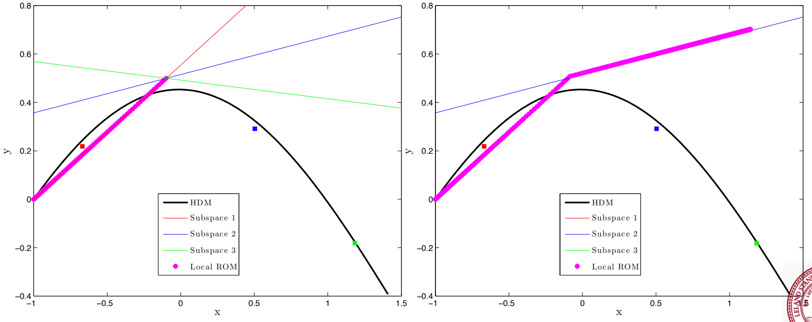
Online Phase: Basis 1

- Select basis whose corresponding *center* $\mathbf{w}_c^i$ is closest to the solution at the previous step $\mathbf{w}_r^{(n-1)}$



Online Phase: Basis 2

- Select basis whose corresponding *center* $\mathbf{w}_c^i$ is closest to the solution at the previous step $\mathbf{w}_r^{(n-1)}$



Online Phase: Basis 2??

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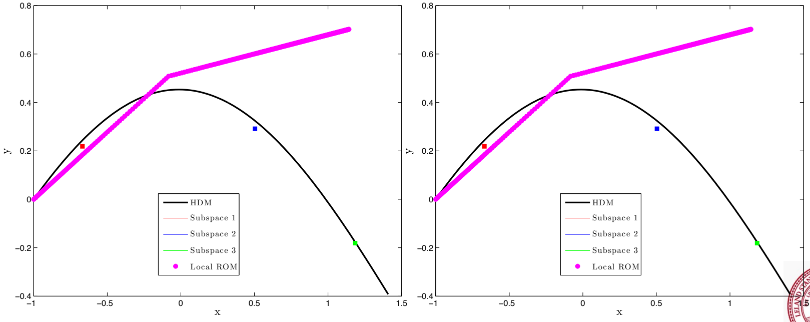

Inconsistency

- Recall the MOR assumption:

$$\mathbf{w}^{(n)} - \mathbf{w}_r^{(n-1)} \approx \mathbf{\Phi}^i \mathbf{y}^{(n)}$$

$$\mathbf{w}^{(n)} - \mathbf{w}^{(switch)} \approx \mathbf{\Phi}^i \sum_{k=switch}^{n} \mathbf{y}^{(k)}$$

where $\mathbf{w}^{(switch)}$ is the most recent state to initiate a switch between bases [Washabaugh et. al. 2012, Zahr et. al. 2014].

- Recall the reduced bases are constructed as

$$\mathbf{\Phi}^i = \text{POD}\left(\mathbf{W}_i - \bar{\mathbf{w}}\mathbf{e}^T\right)$$

- Basis construction consistent with MOR assumption only if $\bar{\mathbf{w}} = \mathbf{w}^{(switch)}$

Solution: Fast Basis Updating

- We seek a reduced basis of the form:

$$\hat{\mathbf{\Phi}}_i = \mathrm{POD}(\mathbf{W}_i - \mathbf{w}^{(switch)}\mathbf{e}^T)$$

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$$
\begin{aligned}
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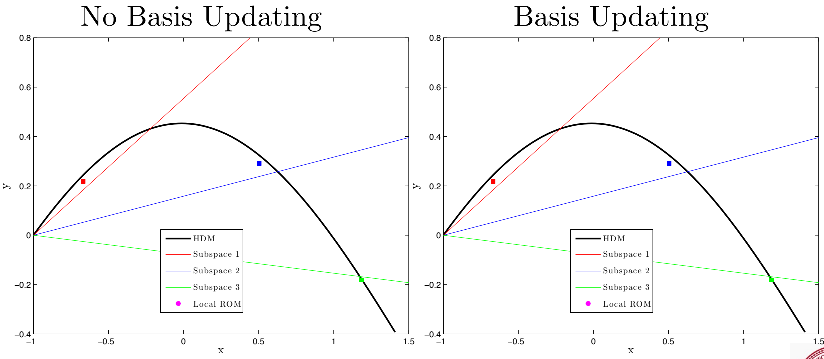
&= \mathrm{POD}(\mathbf{W}_i - \bar{\mathbf{w}}\mathbf{e}^T + (\bar{\mathbf{w}} - \mathbf{w}^{(switch)})\mathbf{e}^T) \\
&= \mathrm{POD}(\hat{\mathbf{W}}_i + (\bar{\mathbf{w}} - \mathbf{w}^{(switch)})\mathbf{e}^T)
\end{aligned}
$$

- $\hat{\mathbf{\Phi}}$ is the (truncated) left singular vectors of a matrix that is a rank-one update of a matrix, $\hat{\mathbf{W}}_i$, whose (truncated) left singular vectors is readily available, $\mathbf{\Phi}_i$.
- Fast updates available [Brand 2006].

Figure : Contrived Example: ROM Solution



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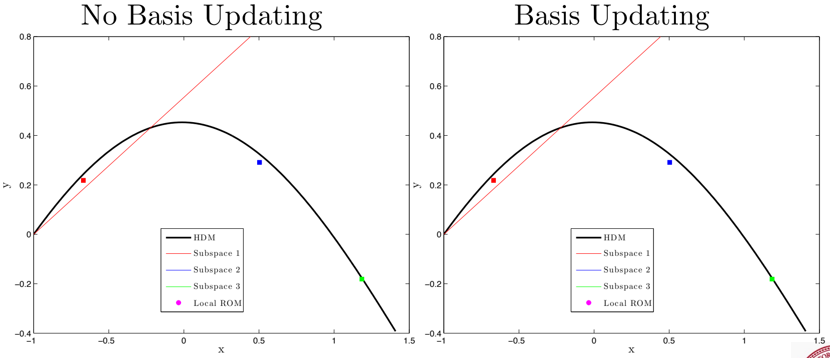

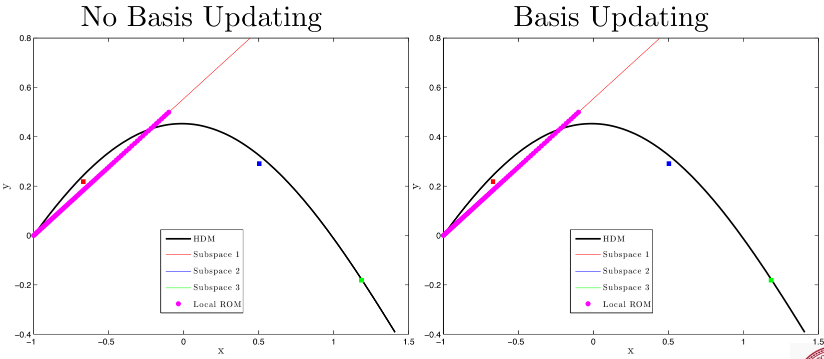

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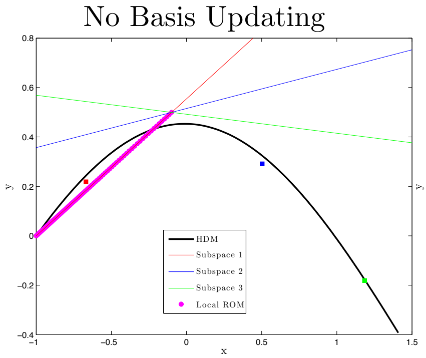

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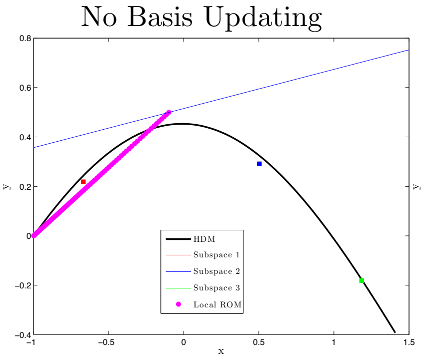

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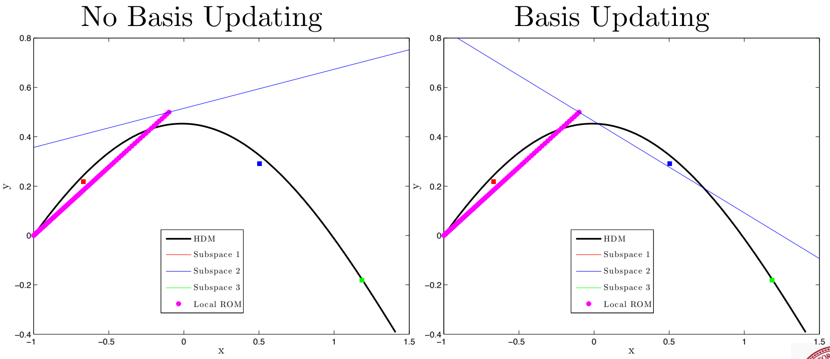

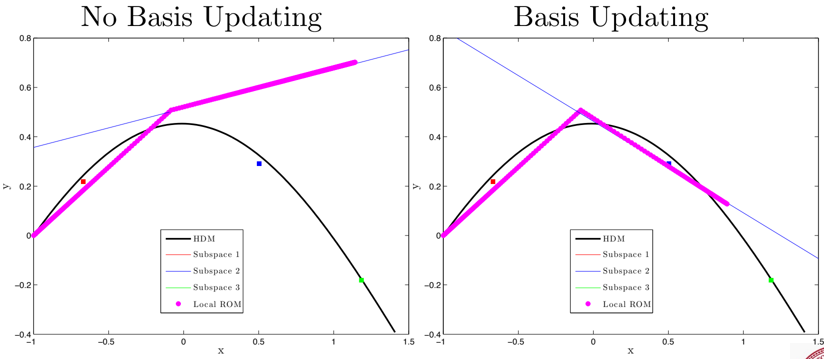

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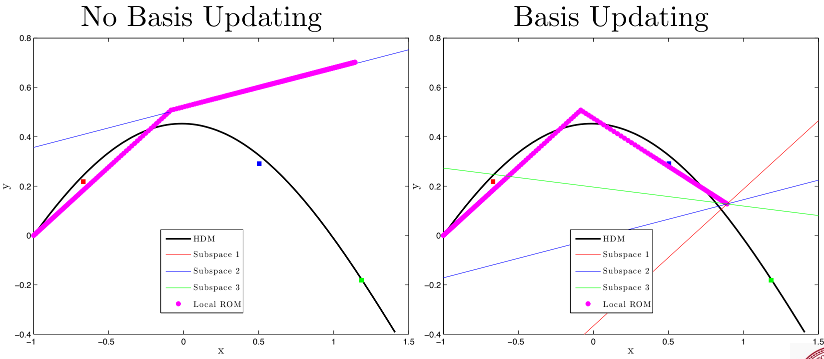

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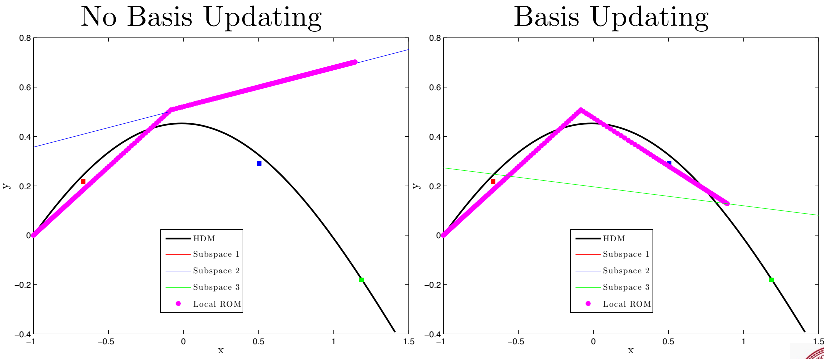

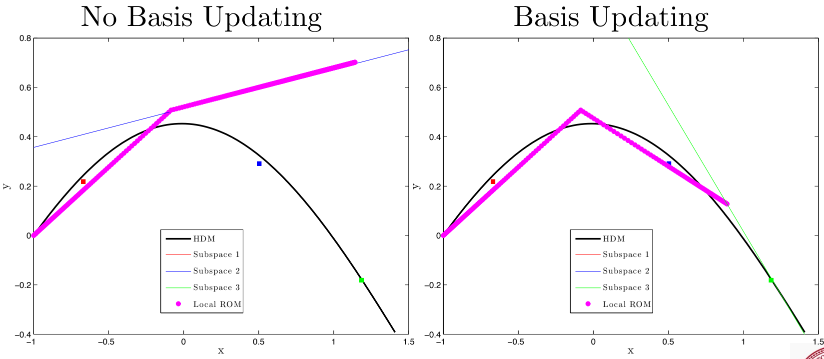

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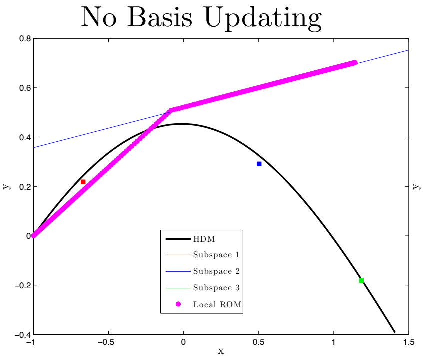

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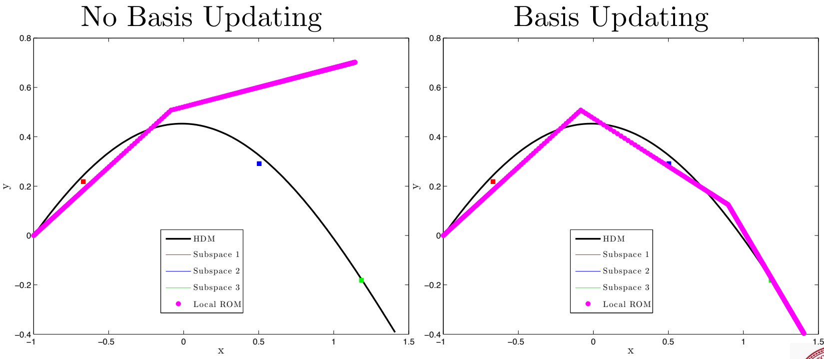
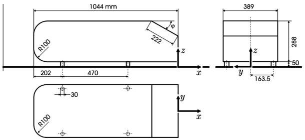
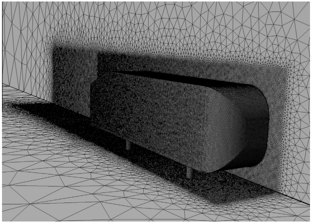


Numerical Example: Ahmed Body

- Benchmark in automotive industry
- Mesh
  - 2,890,434 vertices
  - 17,017,090 tetra
  - 17,342,604 DOF
- CFD
  - Compressible Navier-Stokes
  - DES + Wall func
- Local ROM
  - 5 local bases
  - Size of each ROM: energy criterion



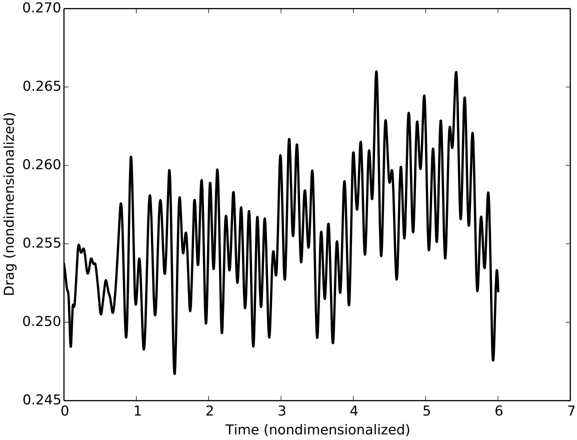
(a) Ahmed Body: Geometry [Ahmed et al 1984]



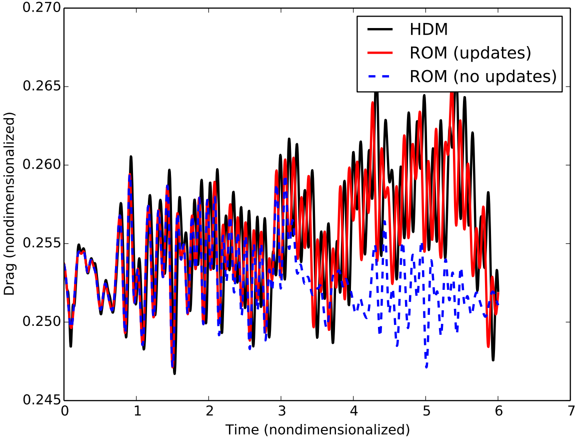
(b) Ahmed Body: Mesh [Carlberg et al 2011]

Ahmed Body Simulation

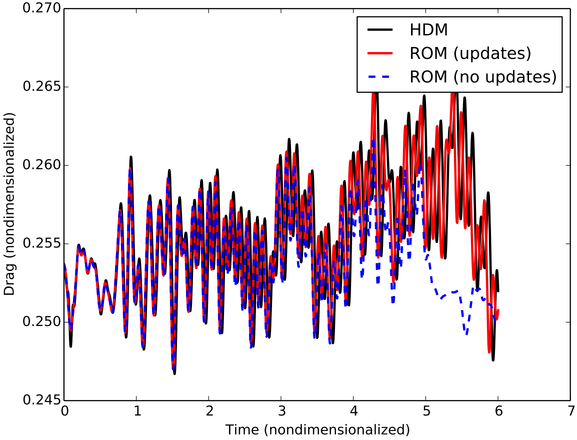
Drag History



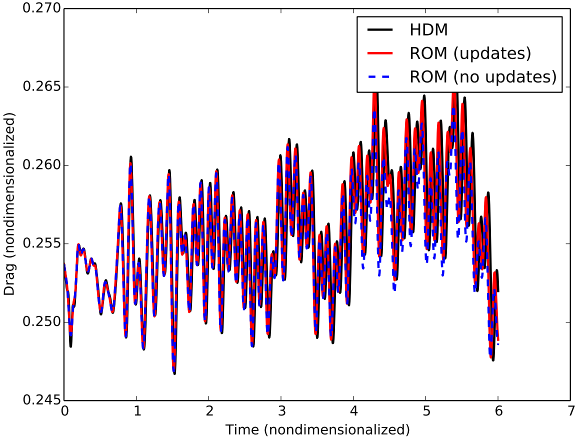
Drag History Comparison: ROM Energy = 99.5%



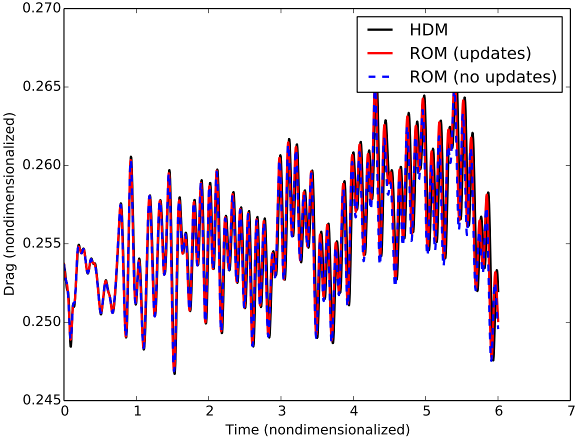
Drag History Comparison: ROM Energy = 99.75%



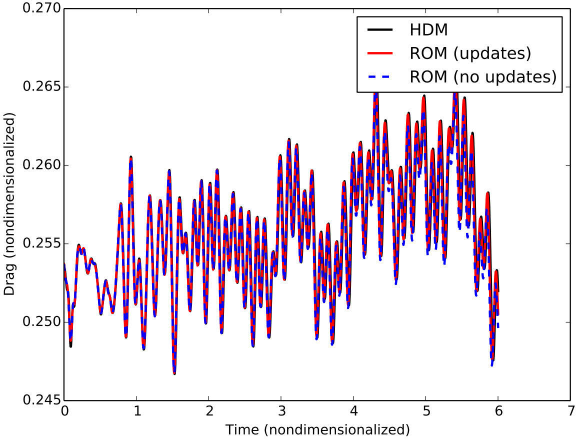
Drag History Comparison: ROM Energy = 99.9%



Drag History Comparison: ROM Energy = 99.95%



Drag History Comparison: ROM Energy = 99.975%



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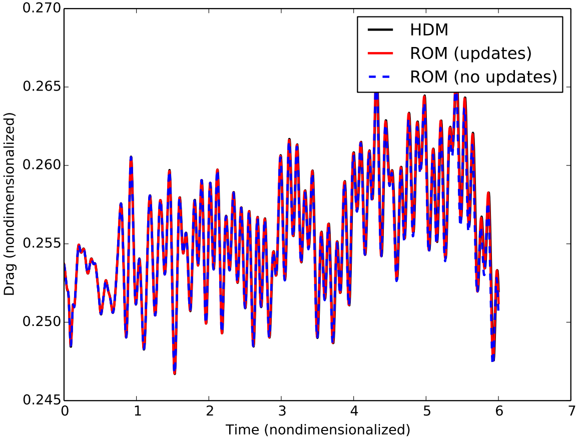
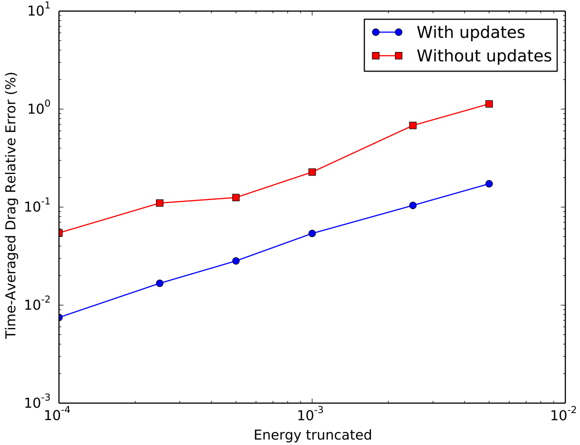
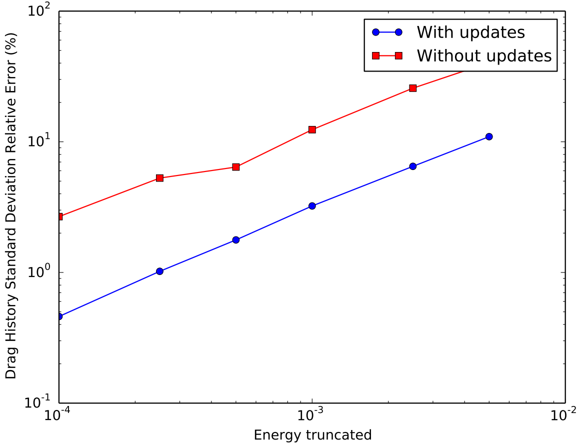


Time-Averaged Drag: Convergence History



Standard Deviation of Drag: Convergence History



Conclusions

- Local model reduction method
  - attractive for problems with distinct solution regimes
  - model reduction assumption and data collection are inconsistent
- Local model reduction with online basis updates
  - addresses inconsistency of local MOR
  - injects "online" data into pre-computed basis
- Applications
  - 3D turbulent flows
  - surrogate in PDE-constrained optimization and uncertainty quantification

Acknowledgements

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